Olympiad Divisibility and Multiples

A Number Theory Guide

Aryan Ayyanger

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Introduction

Divisibility and multiples are foundational pillars in Olympiad number theory. At first glance, the ideas may appear elementary: one number either divides another or it does not. However, in contest mathematics, divisibility becomes a language for detecting hidden structure, proving impossibility, simplifying large expressions, and understanding the architecture of integers.

Divisibility in Olympiad number theory forms the foundation of many advanced concepts. Success relies on mastering formal properties, the Division Algorithm, modular arithmetic, Euclidean methods, prime factorization, and algebraic transformations rather than simply memorizing basic divisibility rules.

Instead of approaching number theory through mechanical steps, this handout focuses on techniques that build intuition and familiarity with problem types. The goal is not just to compute, but also to recognize hidden relationships. A strong contestant learns to ask:

- What structure is hidden inside the expression?
- Can the expression be factored?
- Is there a useful modulus?
- Which prime powers control the divisibility?
- Can a complicated condition be converted into a simpler one?

It is suggested to review the theory for each section and then work on the practice problems. The examples are intentionally written in a teaching style so that the reasoning behind each step is visible.

Basic Definitions

Key Idea

For integers a and b with $b \neq 0$, we say:

$$b \mid a$$

if there exists an integer k such that:

$$a = bk.$$

In words, b divides a.

This definition is simple but extremely powerful. It converts a divisibility statement into an equation involving integers. Many olympiad proofs begin by translating a statement like $b \mid a$ into the form $a = bk$ and then manipulating the resulting equation.

Examples

- $3 \mid 12$ because $12 = 3 \cdot 4$.
- $7 \nmid 20$ because no integer multiplied by 7 equals 20.
- Every nonzero integer divides 0 because $0 = b \cdot 0$.

- 1 divides every integer because $a = 1 \cdot a$.
- If $b \mid a$, then $-b \mid a$ as well, since $a = bk = (-b)(-k)$.

Common Mistake

Do not confuse division with divisibility. The expression $\frac{20}{7}$ exists as a real number, but $7 \nmid 20$ because $\frac{20}{7}$ is not an integer.

Olympiad Insight. Whenever a problem contains a phrase such as "divides," "multiple of," or "leaves a remainder," translate the condition into an equation or a congruence. This mitigates guesswork and gives the problem algebraic structure.

Core Divisibility Properties

Suppose a, b, c are integers.

Key Idea

Important properties:

$$\begin{aligned}
&a \mid b \text{ and } a \mid c \Rightarrow a \mid (b+c),\\
&a \mid b \text{ and } a \mid c \Rightarrow a \mid (b-c),\\
&a \mid b \Rightarrow a \mid bc,\\
&a \mid b \text{ and } b \mid c \Rightarrow a \mid c.
\end{aligned}$$

These properties are the backbone of many olympiad proofs. They allow us to combine known divisible expressions to produce new divisible expressions.

Why These Matter

A common olympiad technique is transforming a difficult expression into a combination of easier divisible expressions.

For example, if we know:

$$7 \mid (a+b)$$

and

$$7 \mid b,$$

then subtracting gives:

$$7 \mid (a+b) - b,$$

so:

$$7 \mid a.$$

That idea appears repeatedly in contest problems.

Linear Combination Principle

A stronger and very useful form is the following.

Key Idea

If:

$$c \mid a \qquad \text{and} \qquad c \mid b,$$

then:

$$c \mid (ax + by)$$

for all integers x and y.

Proof. Since $c \mid a$, there exists an integer m such that $a = cm$. Since $c \mid b$, there exists an integer n such that $b = cn$. Then:

$$ax + by = (cm)x + (cn)y = c(mx + ny).$$

Because $mx + ny$ is an integer, it follows that:

$$c \mid (ax + by).$$

Olympiad Insight. Linear combinations are one of the cleanest ways to eliminate unwanted terms. If two divisibility conditions involve the same variables, try subtracting or multiplying them so that one variable disappears.

Common Mistake

It is false in general that:

$$a \mid bc \Rightarrow a \mid b \text{ or } a \mid c.$$

For example:

$$6 \mid (2 \cdot 3),$$

but:

$$6 \nmid 2 \qquad \text{and} \qquad 6 \nmid 3.$$

This implication becomes valid under additional conditions, such as $\gcd(a, b) = 1$ and $a \mid bc$, in which case $a \mid c$.

Prime Numbers and Factorization

Key Idea

A prime number is an integer greater than 1 with exactly two positive divisors:

$$1 \text{ and itself.}$$

Examples of primes:

$$2, 3, 5, 7, 11, 13, 17.$$

Composite numbers can be written as products of smaller positive integers. Prime numbers are the indivisible building blocks of the integers.

Fundamental Theorem of Arithmetic

Key Idea

Every integer greater than 1 can be written uniquely as a product of prime powers.

Example:

$$360 = 2^3 \cdot 3^2 \cdot 5.$$

Prime factorization is essential because divisibility questions become exponent comparisons. For example:

$$2^a 3^b 5^c \mid 2^x 3^y 5^z$$

exactly when:

$$a \le x, \qquad b \le y, \qquad c \le z.$$

Example. Determine whether:

$$360 \mid 7200.$$

We factor:

$$360 = 2^3 \cdot 3^2 \cdot 5,$$

and:

$$7200 = 2^5 \cdot 3^2 \cdot 5^2.$$

Since every prime exponent in 7200 is at least as large as the corresponding exponent in 360, we conclude:

$$360 \mid 7200.$$

Olympiad Insight. When divisibility involves products, factorials, powers, or least common multiples, prime factorization is often the natural viewpoint. Instead of asking whether one large number divides another, ask whether each prime appears enough times.

Legendre's Formula

Key Idea

For a prime p, the exponent of p in $n!$ is:

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Only finitely many terms are nonzero.

This formula counts how many multiples of p, p^2, p^3, and so on appear in the product:

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

Example. Find the exponent of 2 in 100!.

$$v_2(100!) = \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{8} \right\rfloor + \left\lfloor \frac{100}{16} \right\rfloor + \left\lfloor \frac{100}{32} \right\rfloor + \left\lfloor \frac{100}{64} \right\rfloor.$$

Thus:

$$v_2(100!) = 50 + 25 + 12 + 6 + 3 + 1 = 97.$$

Implications. Factorials grow too quickly to compute directly. Legendre's Formula turns a huge divisibility question into a clean counting problem. It is especially useful when determining whether a number divides $n!$, how many trailing zeros a factorial has, or how prime powers appear inside binomial coefficients.

Greatest Common Divisor and Least Common Multiple

Prime factorization tells us how integers are built. The next natural question is how two integers interact. This leads to greatest common divisors and least common multiples.

Greatest Common Divisor

Key Idea

The greatest common divisor of integers a and b, denoted:

$$\gcd(a, b),$$

is the largest positive integer dividing both numbers.

Example:

$$\gcd(24, 36) = 12.$$

The GCD measures the common structure shared by two integers.

Least Common Multiple

Key Idea

The least common multiple of integers a and b, denoted:

$$\operatorname{lcm}(a, b),$$

is the smallest positive integer divisible by both.

Example:

$$\operatorname{lcm}(12, 18) = 36.$$

The LCM measures the smallest number that contains both divisibility structures.

Important Identity

Key Idea

For positive integers a and b:

$$\gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab.$$

This identity appears frequently in olympiad problems. In prime factorization language, the GCD takes the smaller exponent of each prime, while the LCM takes the larger exponent. For each prime, the smaller exponent plus the larger exponent equals the sum of the exponents in ab.

Example. Let:

$$a = 2^3 \cdot 3^2, \qquad b = 2 \cdot 3^5.$$

Then:

$$\gcd(a,b) = 2^1 \cdot 3^2,$$

and:

$$\operatorname{lcm}(a,b) = 2^3 \cdot 3^5.$$

Multiplying gives:

$$\gcd(a,b)\operatorname{lcm}(a,b) = 2^4 \cdot 3^7 = ab.$$

Euclidean Algorithm

The Euclidean Algorithm is one of the most powerful tools in elementary number theory.

Key Idea

$$\gcd(a,b) = \gcd(b, a-b).$$

More generally,

$$\gcd(a,b) = \gcd(b, a \bmod b).$$

This algorithm is based on the fact that subtracting a multiple of one number from another does not change their common divisors.

Example

Find:

$$\gcd(252, 105).$$

$$\begin{aligned} 252 &= 2(105) + 42, \\ 105 &= 2(42) + 21, \\ 42 &= 2(21) + 0. \end{aligned}$$

Thus:

$$\gcd(252, 105) = 21.$$

Human Insight

Many students try to factor large numbers immediately. Olympiad problems often reward using structure first. The Euclidean Algorithm avoids unnecessary computation and reveals patterns quickly.

The Euclidean Algorithm is also useful because it can be reversed. Reversing the steps expresses the GCD as an integer linear combination of the original numbers. This idea leads to Bezout's identity:

$$\gcd(a, b) = ax + by$$

for some integers x and y.

Olympiad Insight. Whenever a problem involves $\gcd(a, b)$ and the expressions $a+b$, $a-b$, $ka+b$, or remainders, try applying the Euclidean Algorithm. It often reduces a complicated expression to a much smaller one.

Divisibility Tests

Divisibility tests are not merely shortcuts; they are examples of modular arithmetic in base 10. Understanding why they work is more valuable than memorizing the rules.

Divisibility by 2

A number is divisible by 2 if its last digit is even. This works because powers of 10 beyond the units digit are divisible by 2.

Divisibility by 3

A number is divisible by 3 if the sum of its digits is divisible by 3.

Example:

$$813 \to 8 + 1 + 3 = 12.$$

Since 12 is divisible by 3, so is 813.

The reason is:

$$10 \equiv 1 \pmod 3,$$

so every power of 10 is also congruent to 1 modulo 3.

Divisibility by 9

A number is divisible by 9 if its digit sum is divisible by 9. This works because:

$$10 \equiv 1 \pmod 9.$$

Divisibility by 11

Take alternating sums of digits.

Example:

$$2728.$$

Compute:

$$(2 + 2) - (7 + 8) = -11.$$

Since -11 is divisible by 11, the number is divisible by 11.

This works because:

$$10 \equiv -1 \pmod{11}.$$

Thus the powers of 10 alternate between 1 and -1 modulo 11.

Olympiad Insight. Digit problems are usually modular arithmetic problems in disguise. If a problem asks about a number written in decimal form, think about powers of 10 modulo the divisor.

Modular Arithmetic

Key Idea

We write:

$$a \equiv b \pmod{n}$$

if:

$$n \mid (a - b).$$

Modular arithmetic is essentially arithmetic with remainders. It allows us to replace large numbers with smaller representatives while preserving divisibility information.

Examples

$$17 \equiv 2 \pmod{5}$$

because:

$$17 - 2 = 15,$$

which is divisible by 5.

Similarly:

$$-1 \equiv 4 \pmod{5}.$$

Key Operations

If:

$$a \equiv b \pmod{n}$$

and:

$$c \equiv d \pmod{n},$$

then:

$$\begin{aligned} a + c &\equiv b + d \pmod{n}, \\ a - c &\equiv b - d \pmod{n}, \\ ac &\equiv bd \pmod{n}. \end{aligned}$$

These rules allow ordinary algebra to be performed with remainders.

Power Cycles

Powers often repeat modulo n. This is one of the main reasons modular arithmetic is so useful.

Example. Find:

$$2^{100} \pmod 7.$$

Since:

$$2^3 = 8 \equiv 1 \pmod 7,$$

we have:

$$2^{99} = (2^3)^{33} \equiv 1^{33} \equiv 1 \pmod 7.$$

Therefore:

$$2^{100} \equiv 2 \pmod 7.$$

Fermat's Little Theorem

Key Idea

If p is prime, then for any integer a:

$$a^p \equiv a \pmod p.$$

If additionally $\gcd(a, p) = 1$, then:

$$a^{p-1} \equiv 1 \pmod p.$$

Example. Compute:

$$3^{100} \pmod 7.$$

Since:

$$3^6 \equiv 1 \pmod 7,$$

and:

$$100 = 6(16) + 4,$$

we get:

$$3^{100} \equiv 3^4 = 81 \equiv 4 \pmod 7.$$

Euler's Totient Theorem

Key Idea

If:

$$\gcd(a, n) = 1,$$

then:

$$a^{\phi(n)} \equiv 1 \pmod n,$$

where $\phi(n)$ counts the positive integers less than or equal to n that are relatively prime to n.

Euler's theorem generalizes Fermat's Little Theorem from prime moduli to composite moduli.

Example. Since:

$$\phi(10) = 4,$$

and:

$$\gcd(3, 10) = 1,$$

we have:

$$3^4 \equiv 1 \pmod{10}.$$

Therefore:

$$3^{100} = (3^4)^{25} \equiv 1 \pmod{10}.$$

Olympiad Insight. Use Fermat's Little Theorem or Euler's Theorem when a problem involves high powers and a modulus. Before applying Euler's theorem, always check the required condition:

$$\gcd(a, n) = 1.$$

Classic Olympiad Techniques

Working Backwards

Instead of directly proving divisibility, manipulate expressions until familiar factors appear.

Example:

$$n^3 - n.$$

Factor:

$$n(n^2 - 1) = n(n-1)(n+1).$$

This is the product of three consecutive integers.

Among any three consecutive integers:

- one is divisible by 3,
- at least one is divisible by 2.

Thus:

$$6 \mid n(n-1)(n+1).$$

Therefore:

$$6 \mid (n^3 - n).$$

Pattern Recognition. Expressions like:

$$n^3 - n, \quad n^5 - n, \quad a^n - b^n$$

often suggest factorization. Before using modular arithmetic or checking cases, always ask whether the expression has a standard algebraic factorization.

Bounding and Contradiction

Bounding involves placing quantities within upper or lower limits, while contradiction proves a statement by assuming the opposite and deriving an impossibility.

In olympiad number theory, bounding and contradiction are especially useful when integer restrictions are present. A real-valued quantity may lie in an interval, but an integer cannot lie between two consecutive integers. This simple idea is often decisive.

1. Definitions

Definition 9.1 (Upper Bound)**.** *A number M is an* ***upper bound*** *for a quantity x if:*

$$x \leq M.$$

Definition 9.2 (Lower Bound)**.** *A number m is a* ***lower bound*** *for a quantity x if:*

$$x \geq m.$$

Definition 9.3 (Proof by Contradiction)**.** *To prove a statement P, assume that P is false and show that this assumption leads to a contradiction.*

2. Common Bounding Ideas

- Compare expressions with known inequalities:

$$a^2 + b^2 \geq 2ab.$$

- Use extremal values, such as the largest or smallest element.
- Count the same object in two different ways.
- Replace complicated expressions with simpler upper or lower estimates.
- Use integer constraints: if an integer satisfies

$$3 < n < 4,$$

  then no such integer exists.

Olympiad Insight. Bounding becomes especially strong when combined with equality conditions. If an inequality forces equality, then the equality case often reveals the only possible solution.

3. Example: Bounding

Problem 9.1. *Show that for positive real numbers a, b:*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Solution. Since:

$$(a-b)^2 \geq 0,$$

we expand:

$$a^2 - 2ab + b^2 \geq 0.$$

Adding $4ab$ gives:

$$a^2 + 2ab + b^2 \geq 4ab.$$

Thus:

$$(a+b)^2 \geq 4ab.$$

Because $a, b > 0$, both sides are nonnegative, so taking square roots gives:

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Equality occurs exactly when:

$$a = b.$$

□

4. Example: Contradiction

Problem 9.2. *Prove that $\sqrt{2}$ is irrational.*

Solution. Assume for contradiction that:

$$\sqrt{2} = \frac{a}{b}$$

for integers a, b with $\gcd(a, b) = 1$.

Squaring:

$$2 = \frac{a^2}{b^2},$$

so:

$$a^2 = 2b^2.$$

Thus a^2 is even, implying a is even. Write:

$$a = 2k.$$

Substituting:

$$(2k)^2 = 2b^2,$$

which gives:

$$4k^2 = 2b^2,$$

and therefore:

$$b^2 = 2k^2.$$

Hence b is also even.

But then both a and b are divisible by 2, contradicting:

$$\gcd(a, b) = 1.$$

Therefore $\sqrt{2}$ is irrational. □

5. Olympiad Problems

Problem 9.3. *Let x, y, z be positive integers satisfying:*

$$x + y + z = 10.$$

Find the maximum possible value of:

$$xyz.$$

Solution. By AM-GM:

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}.$$

Thus:

$$\frac{10}{3} \geq \sqrt[3]{xyz}.$$

Cubing:

$$xyz \leq \left(\frac{10}{3}\right)^3 \approx 37.03.$$

Since xyz is an integer:

$$xyz \leq 37.$$

Now check triples closest to equality in AM-GM. Equality would occur near:

$$x = y = z = \frac{10}{3}.$$

The closest positive integer triple is:

$$3, 3, 4,$$

which gives:

$$xyz = 36.$$

Hence the maximum value is:

$$\boxed{36}.$$

□

Problem 9.4. *Prove that there do not exist integers a, b such that:*

$$a^2 - b^2 = 1$$

and both $a, b > 1$.

Solution. Factor:

$$a^2 - b^2 = (a - b)(a + b) = 1.$$

Since $a, b > 1$, both factors are positive integers.

The only positive factorization of 1 is:

$$1 \cdot 1.$$

Thus:

$$a - b = 1, \qquad a + b = 1.$$

Adding:

$$2a = 2,$$

so:

$$a = 1,$$

contradicting $a > 1$.

Therefore no such integers exist. □

6. Typical Olympiad Strategies

1. Assume an extremal configuration exists.
2. Derive upper and lower bounds for the same quantity.
3. Force equality conditions.
4. Use parity or divisibility restrictions.
5. Assume the opposite statement and search for an impossibility.

7. Practice Problems

1. Prove that among any $n+1$ integers, two have the same remainder modulo n.
2. Show that if:
$$a+b+c=1,$$
then:
$$a^2+b^2+c^2 \geq \frac{1}{3}.$$
3. Prove that there is no rational number whose square is 3.
4. Let x, y be positive reals with:
$$xy=1.$$
Find the minimum value of:
$$x+y.$$
5. Prove that in any group of six people, either three mutually know each other or three mutually do not know each other.

Worked Example Problems

Example

Problem 1.
Prove that for every integer n:
$$5 \mid (n^5-n).$$

Solution.
Factor:
$$n^5-n=n(n^4-1).$$
Continue:
$$=n(n^2-1)(n^2+1).$$
Then:
$$=n(n-1)(n+1)(n^2+1).$$

This factorization is useful, but it does not immediately show divisibility by 5 in every case. A cleaner method uses modular arithmetic.

Any integer is congruent to one of:
$$0,1,2,3,4 \pmod 5.$$
Checking each case:
$$0^5 \equiv 0, \quad 1^5 \equiv 1, \quad 2^5=32 \equiv 2, \quad 3^5=243 \equiv 3, \quad 4^5 \equiv 4 \pmod 5.$$
Thus:
$$n^5 \equiv n \pmod 5.$$
Therefore:
$$n^5-n \equiv 0 \pmod 5,$$
so:
$$5 \mid (n^5-n).$$

Key Observation. This is a special case of Fermat's Little Theorem:

$$n^5 \equiv n \pmod 5.$$

Example

Problem 2.
Find all integers n such that:

$$12 \mid n^2 - 4.$$

Solution.
Factor:

$$n^2 - 4 = (n-2)(n+2).$$

We require divisibility by:

$$12 = 3 \cdot 4.$$

It is often easier to analyze divisibility by 3 and by 4 separately.
Modulo 3:

$$n^2 - 4 \equiv n^2 - 1 \pmod 3.$$

Thus:

$$n^2 \equiv 1 \pmod 3,$$

so:

$$n \not\equiv 0 \pmod 3.$$

Modulo 4:

$$n^2 - 4 \equiv n^2 \pmod 4.$$

Thus:

$$n^2 \equiv 0 \pmod 4,$$

which means:

$$n \equiv 0 \text{ or } 2 \pmod 4.$$

Combining these conditions modulo 12, we get:

$$n \equiv 2, 4, 8, 10 \pmod{12}.$$

Takeaway

Notice that olympiad problems rarely require blind computation. They reward recognizing algebraic structure. In this example, the key moves were:

- factor the expression,
- split divisibility by 12 into divisibility by 3 and 4,
- combine the resulting congruence conditions.

Common Patterns

Difference of Squares

$$a^2 - b^2 = (a-b)(a+b).$$

This identity is especially useful when two squares differ by a small number.

Difference of Cubes

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

Sum of Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Sophie Germain Identity

Key Idea

$$a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2).$$

For example:

$$x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2).$$

Recognizing these instantly saves enormous amounts of time in contests.

Pattern Recognition. When a problem contains expressions such as:

$$a^2 - b^2, \quad a^3 - b^3, \quad a^3 + b^3, \quad a^4 + 4b^4,$$

try factorization before trying modular cases. Many divisibility problems are designed so that the main difficulty is seeing the hidden factorization.

Advanced Olympiad Ideas

Infinite Descent

Infinite descent proves impossibility by constructing endlessly smaller positive integers.

The typical structure is:

1. Assume a positive integer solution exists.
2. Choose the smallest such solution.
3. Use the equation to construct an even smaller positive solution.
4. This contradicts minimality.

Olympiad Insight. Infinite descent is a refined form of contradiction. It is especially powerful in problems involving squares, rationality, divisibility chains, and equations where every solution forces a smaller solution.

Valuation

The exponent of a prime p in a factorization is denoted:

$$v_p(n).$$

Example:

$$v_2(48) = 4$$

because:

$$48 = 2^4 \cdot 3.$$

Valuation arguments are powerful in advanced olympiad divisibility because they allow contestants to compare prime exponents precisely.

Example. If:

$$a^2 \mid b^3,$$

then for every prime p:

$$2v_p(a) \le 3v_p(b).$$

This converts a divisibility condition into inequalities about exponents.

Chinese Remainder Theorem

If congruences involve coprime moduli, they can often be combined into a single congruence.

Key Idea

If $\gcd(m, n) = 1$, then a system:

$$x \equiv a \pmod{m}, \qquad x \equiv b \pmod{n}$$

has a unique solution modulo mn.

Example. Solve:

$$x \equiv 2 \pmod{3}, \qquad x \equiv 3 \pmod{5}.$$

Checking numbers congruent to 2 modulo 3:

$$2, 5, 8, 11, 14, \ldots$$

The number 8 satisfies:

$$8 \equiv 3 \pmod{5}.$$

Thus:

$$x \equiv 8 \pmod{15}.$$

Problem Set

Introductory Problems

1. Prove that:
$$3 \mid (n^3 - n)$$
for every integer n.

2. Find:
$$\gcd(144, 252).$$

3. Determine whether:
$$11 \mid 918273.$$

4. Prove that the product of two consecutive integers is always even.

5. Show that if:
$$a \mid b$$
and:
$$a \mid c,$$
then:
$$a \mid (5b - 2c).$$

Intermediate Problems

6. Prove that:
$$30 \mid (n^5 - n)$$
for every integer n.

7. Find all integers satisfying:
$$7 \mid n^2 + 3.$$

8. Prove:
$$\gcd(a + b, a - b) \mid 2a.$$

9. Show that among any $n + 1$ integers, two have the same remainder modulo n.

10. Determine all integers n such that:
$$n \mid (n + 5)(n + 7).$$

11. Prove that the only even prime number is 2.

Advanced Problems

11. Prove that if:

$$2^n - 1$$

is prime, then n is prime.

12. Show that:

$$\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1.$$

13. Prove that there are infinitely many integers not representable as:

$$a^2 + b^2.$$

14. Let:

$$a + b + c = 0.$$

Prove that:

$$a^3 + b^3 + c^3$$

is divisible by:

$$3abc.$$

15. Find all positive integers n such that:

$$n^2 + 2 \mid n!.$$

Problem-Solving Guidance

For the introductory problems, focus on direct use of definitions, factorization, and small modular arguments.

For the intermediate problems, expect to combine multiple ideas. For example, proving:

$$30 \mid n^5 - n$$

requires divisibility by 2, 3, and 5.

For the olympiad-level problems, the main challenge is usually not computation. The difficulty is finding the correct viewpoint: contradiction, factorization, Euclidean structure, valuation, or modular order.

Strategies for Olympiad Success

- Look for factorization before computation.
- Translate divisibility into modular arithmetic.
- Test small cases to identify patterns.
- Ask whether parity matters.
- Use prime factorization aggressively.
- Track prime exponents when factorials or powers appear.

- Use the Euclidean Algorithm when GCD expressions appear.
- Avoid random guessing once a structure appears.

Common Mistake

Many olympiad students lose points not because the problem is impossible, but because they abandon structure too early and begin brute forcing calculations.

Section Strategy Summary

When facing a divisibility problem, try the following sequence:

1. Check small moduli such as 2, 3, 4, 5, 7, and 11.
2. Factor the expression if possible.
3. Rewrite the divisibility statement using definitions or modulus.
4. Identify whether prime factorization or valuations are needed.
5. Look for a theorem such as Fermat's Little Theorem, Euler's Theorem, or the Chinese Remainder Theorem.

Closing Thoughts

Divisibility is a building block for more complex topics in olympiad number theory. Olympiad problems often hide an elegant structure beneath intimidating algebra, and require the use of divisibility rules for preliminary steps.

Improvement comes from pattern recognition and repeated exposure. Familiarizing oneself with structures like factorization and congruences builds intuition needed for competition competitiveness.

Resources and References

1. Art of Problem Solving - `https://artofproblemsolving.com/?srsltid=AfmBOopV5ZIDR_i1thFxW75wFTQC3mt-NVH7PEoFs4xxWrjtubG1UMeF`
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5. Sohil Rathi - `https://www.youtube.com/c/SohilRathi`