Olympiad Counting and Combinatorics

Fundamental Counting, Permutations, Inclusion–Exclusion, and Double Counting

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Contents

Introduction

Combinatorics is the study of counting, arranging, selecting, and organizing discrete objects. At first glance, it may seem like a collection of formulas:

$$n!, \qquad \binom{n}{k}, \qquad 2^n.$$

However, olympiad combinatorics is not mainly about memorizing formulas. It is about recognizing structure.

In a contest problem, the most difficult part is often not the arithmetic. The difficult part is deciding what is being counted, whether order matters, whether objects are repeated, and whether some objects are accidentally being counted more than once.

A strong contestant learns to ask:

- What are the objects being counted?
- When are two objects considered different?
- Does order matter?
- Are there restrictions or forbidden arrangements?
- Is the complement easier to count?
- Is there overlap between cases?
- Can the same object be counted in two different ways?

This handout develops several core olympiad counting techniques: fundamental counting principles, permutations, combinations, inclusion–exclusion, derangements, and double counting.

Key Idea

The central goal of combinatorics is:

$$\boxed{\text{Count every valid object exactly once.}}$$

Almost every mistake in combinatorics comes from violating this principle. A solution may overcount, undercount, or count invalid objects.

The Philosophy of Counting

Before using a formula, one should identify the structure of the problem.

Counting Objects

A counting problem always involves a collection of objects. These objects might be:

- strings,
- subsets,

- arrangements,
- committees,
- paths,
- functions,
- colorings,
- graphs,
- integer solutions.

Identifying the object is the first step.

Example

How many binary strings of length 6 contain exactly two 1's?

Solution. A valid string is determined by the positions of the two 1's.

Thus we choose 2 positions out of 6:

$$\binom{6}{2} = 15.$$

□

The original problem is about strings, but the solution is about choosing positions.

When Are Two Objects Different?

Suppose three students are chosen for a committee. The group:

$$(A, B, C)$$

is the same committee as:

$$(B, C, A).$$

Order does not matter.

But if gold, silver, and bronze medals are awarded, then:

$$(A, B, C)$$

and:

$$(B, C, A)$$

represent different outcomes.

Common Mistake

Before choosing a formula, decide whether order matters.

The Addition Principle

Key Idea

If a task can be completed in one of several non-overlapping ways, then the total number of ways is the sum of the number of ways in each case.

If A and B are disjoint sets, then:

$$|A \cup B| = |A| + |B|.$$

Example

Example

How many positive integers less than 100 are either one-digit or two-digit?

Solution. There are 9 one-digit positive integers and 90 two-digit positive integers.

These categories do not overlap, so:

$$9 + 90 = 99.$$

□

Why Non-Overlap Matters

If categories overlap, simple addition fails.

For example, the number of integers from 1 to 100 divisible by 2 or 3 is not:

$$50 + 33.$$

This counts multiples of 6 twice.

This issue leads naturally to inclusion–exclusion.

The Multiplication Principle

Key Idea

If a process occurs in stages, multiply the number of choices at each stage.

Example

Example

How many license plates consist of three letters followed by three digits?

Solution. Each of the three letter positions has 26 choices, and each of the three digit positions has 10 choices.

Thus:

$$26^3 \cdot 10^3.$$

□

Dependent Choices

Choices may depend on earlier choices.

Example

How many three-digit numbers have distinct digits?

Solution. The first digit cannot be 0, so it has 9 choices.

The second digit can be any digit except the first, giving 9 choices.

The third digit can be any digit except the first two, giving 8 choices.

Thus:

$$9 \cdot 9 \cdot 8.$$

□

Common Mistake

Do not assume every stage has the same number of choices. Restrictions often change later choices.

Complementary Counting

Sometimes it is easier to count what is not wanted.

Key Idea

$$\text{Desired} = \text{Total} - \text{Undesired}.$$

Example

How many binary strings of length 10 contain at least one 1?

Solution. There are:

$$2^{10}$$

binary strings total.

Only one string contains no 1's:

$$0000000000.$$

Therefore the answer is:

$$2^{10} - 1 = 1023.$$

□

Olympiad Insight. Phrases like "at least one," "not all," "avoid," or "no two" often suggest complementary counting.

Casework

Casework means dividing a problem into non-overlapping cases.

Good casework must satisfy:

1. Every valid object appears in at least one case.
2. No valid object appears in more than one case.

Example

How many positive integers less than 100 have digit sum 5?

Solution. There is one one-digit number:

$$5.$$

For two-digit numbers, the possibilities are:

$$14, 23, 32, 41, 50.$$

Thus the total is:

$$6.$$

□

Common Mistake

Bad casework is one of the most common sources of wrong combinatorics answers. Cases must be exhaustive and non-overlapping.

Permutations

A permutation is an arrangement of objects in a particular order.

Key Idea

The number of ways to arrange n distinct objects in a line is:

$$n!.$$

Example

Example

How many ways can 7 distinct books be arranged on a shelf?

Solution. Since all books are distinct and order matters:

$$7!.$$

□

Partial Permutations

Sometimes only some objects are arranged.

Example

How many ways can gold, silver, and bronze medals be awarded to 10 students?

Solution. Gold has 10 choices, silver has 9, and bronze has 8.
Thus:

$$10 \cdot 9 \cdot 8.$$

□

In general, the number of ordered selections of k objects from n distinct objects is:

$$\frac{n!}{(n-k)!}.$$

Permutations with Repeated Objects

If objects repeat, ordinary factorial counting overcounts.

Key Idea

If n objects contain repeated groups of sizes $a_1, a_2, \dots, a_r$, then the number of distinct arrangements is:

$$\frac{n!}{a_1!a_2! \cdots a_r!}.$$

Example

How many distinct arrangements of the letters in BANANA exist?

Solution. BANANA has 6 letters:

- A appears 3 times,
- N appears 2 times,
- B appears 1 time.

Thus:

$$\frac{6!}{3!2!}.$$

□

Example: MISSISSIPPI

Example

How many distinct arrangements of the letters in MISSISSIPPI exist?

Solution. MISSISSIPPI has 11 letters:

- I appears 4 times,
- S appears 4 times,
- P appears 2 times,
- M appears 1 time.

Therefore:

$$\frac{11!}{4!4!2!}.$$

□

Olympiad Insight. Repeated objects create symmetry. Dividing by repeated factorials removes the overcount.

Circular Permutations

In a circle, rotations are usually considered the same.

Key Idea

The number of circular arrangements of n distinct objects is:

$$(n-1)!.$$

This works because one object can be fixed, and the remaining $n-1$ objects can be arranged around it.

Example

How many ways can 8 people sit around a circular table?

Solution. Fix one person. Arrange the remaining 7 people:

$$7!.$$

□

Common Mistake

Circular table problems usually identify rotations but not reflections. Necklace problems may identify both rotations and reflections.

Block Method

The block method is used when objects must stay together.

Example

How many ways can 6 people stand in a line if Alice and Bob must stand together?

Solution. Treat Alice and Bob as one block.

Now there are 5 objects to arrange:

$$(AB), C, D, E, F.$$

These can be arranged in:

$$5!$$

ways.

Inside the block, Alice and Bob can be ordered in:

$$2!$$

ways.

Thus:

$$5! \cdot 2!.$$

□

Gap Method

The gap method is used when certain objects must not be adjacent.

Example

How many ways can 5 boys and 3 girls stand in a line if no two girls stand next to each other?

Solution. First arrange the boys:

$$5!.$$

The boys create 6 gaps:

$$\_B\_B\_B\_B\_B\_.$$

Choose 3 of these 6 gaps for the girls:

$$\binom{6}{3}.$$

Arrange the girls:

$$3!.$$

Thus:

$$5!\binom{6}{3}3!.$$

□

Olympiad Insight. For no-adjacency problems, first arrange the separating objects, then place the restricted objects into gaps.

Combinations

A combination is a selection where order does not matter.

Key Idea

The number of ways to choose k objects from n distinct objects is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example

How many committees of 4 can be chosen from 12 students?

Solution. Since order does not matter:

$$\binom{12}{4}.$$

□

Committee with Roles

Example

A club has 12 members. How many ways can a president, vice president, secretary, and treasurer be chosen?

Solution. Roles are distinct, so order matters:

$$12 \cdot 11 \cdot 10 \cdot 9.$$

□

Common Mistake

Choosing a committee is not the same as assigning positions.

Binomial Coefficients

Binomial coefficients count subsets.

$$\binom{n}{k}$$

counts the number of k-element subsets of an n-element set.

Symmetry Identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

Choosing k elements to include is the same as choosing $n-k$ elements to exclude.

Pascal's Identity

Key Idea

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Solution. Count committees of size k from n people. Focus on one particular person, Alex.

A committee either excludes Alex or includes Alex.

If it excludes Alex:

$$\binom{n-1}{k}.$$

If it includes Alex:

$$\binom{n-1}{k-1}.$$

Thus:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

□

Stars and Bars

Stars and Bars counts nonnegative integer solutions.

Key Idea

The number of nonnegative integer solutions to:

$$x_1 + x_2 + \cdots + x_k = n$$

is:

$$\binom{n+k-1}{k-1}.$$

Example

How many nonnegative integer solutions are there to:

$$x + y + z = 12?$$

Solution. Here $n = 12$ and $k = 3$.

Thus:

$$\binom{12+3-1}{3-1} = \binom{14}{2}.$$

□

Positive Solutions

The number of positive integer solutions to:

$$x_1 + x_2 + \cdots + x_k = n$$

is:

$$\binom{n-1}{k-1}.$$

Example

How many positive integer solutions are there to:

$$x + y + z = 12?$$

Solution. Let:

$$a = x - 1, \quad b = y - 1, \quad c = z - 1.$$

Then:

$$a + b + c = 9.$$

The number of nonnegative solutions is:

$$\binom{11}{2}.$$

□

Inclusion–Exclusion Principle

The Inclusion–Exclusion Principle handles overlapping sets.

Key Idea

For two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Key Idea

For three finite sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example

Example

How many integers from 1 to 100 are divisible by 2 or 3?

Solution. Let A be the set of multiples of 2, and B the set of multiples of 3.

$$|A| = \left\lfloor \frac{100}{2} \right\rfloor = 50.$$

$$|B| = \left\lfloor \frac{100}{3} \right\rfloor = 33.$$

The overlap consists of multiples of 6:

$$|A \cap B| = \left\lfloor \frac{100}{6} \right\rfloor = 16.$$

Therefore:

$$|A \cup B| = 50 + 33 - 16 = 67.$$

□

Olympiad Insight. When simple addition double-counts overlap, use inclusion–exclusion.

PIE with Three Sets

Example

How many integers from 1 to 1000 are divisible by at least one of $2, 3, 5$?

Solution. Let:

$$A = \{n : 2 \mid n\}, \quad B = \{n : 3 \mid n\}, \quad C = \{n : 5 \mid n\}.$$

Then:

$$|A| = 500, \quad |B| = 333, \quad |C| = 200.$$

Pairwise intersections:

$$|A \cap B| = \left\lfloor \frac{1000}{6} \right\rfloor = 166,$$

$$|A \cap C| = \left\lfloor \frac{1000}{10} \right\rfloor = 100,$$

$$|B \cap C| = \left\lfloor \frac{1000}{15} \right\rfloor = 66.$$

Triple intersection:

$$|A \cap B \cap C| = \left\lfloor \frac{1000}{30} \right\rfloor = 33.$$

Thus:

$$500 + 333 + 200 - 166 - 100 - 66 + 33 = 734.$$

□

Derangements

A derangement is a permutation with no fixed points.

For example, if n people each have a hat, a derangement is a way to return hats so that no person receives their own hat.

Key Idea

The number of derangements of n objects is:

$$!n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

Derivation Using Inclusion–Exclusion

Let A_i be the set of permutations where object i is fixed.

We want to count permutations where none of the A_i occur:

$$n! - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Using PIE:

$$!n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0!.$$

This simplifies to:

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Example

Example

How many derangements are there of 4 objects?

Solution.

$$!4 = 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right).$$

Thus:

$$!4 = 24\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 9.$$

□

Double Counting

Double counting means counting the same set of objects in two different ways.

Key Idea

If two expressions count the same collection of objects, they must be equal.

First Example

Example

Prove:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Solution. Count subsets of an n-element set.

Method 1: Each element is either included or excluded. Thus there are:

$$2^n$$

subsets.

Method 2: Count by subset size. There are:

$$\binom{n}{k}$$

subsets of size k.

Summing over all k:

$$\sum_{k=0}^{n} \binom{n}{k}.$$

Therefore:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

□

Committee Double Counting

Example

Prove:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Solution. Count the number of ways to choose a committee of k people and then choose a chairperson from that committee.

Method 1: Choose the committee first:

$$\binom{n}{k}.$$

Then choose the chair from the k members:

$$k.$$

This gives:

$$k\binom{n}{k}.$$

Method 2: Choose the chairperson first:

$$n.$$

Then choose the other $k-1$ committee members:

$$\binom{n-1}{k-1}.$$

This gives:

$$n\binom{n-1}{k-1}.$$

Therefore:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

□

Counting Marked Subsets

Example

Prove:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Solution. Count pairs (S, a) where:

$$S \subseteq \{1, 2, \ldots, n\}$$

and:

$$a \in S.$$

Method 1: Choose S first. If $|S| = k$, there are k choices for a.
This gives:

$$\sum_{k=0}^{n} k\binom{n}{k}.$$

Method 2: Choose a first:

$$n$$

choices.

Then choose the rest of S. The element a must be included, and each of the remaining $n-1$ elements may be included or excluded.

This gives:

$$n2^{n-1}.$$

Therefore:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

□

Handshake Lemma

The Handshake Lemma is a classic double counting result in graph theory.

Key Idea

In any finite graph:

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Each edge contributes 1 to the degree of each of its two endpoints, so every edge is counted twice.

Consequence

Every finite graph has an even number of vertices with odd degree.

Solution. The sum of all degrees is even because it equals $2|E|$.

The sum of all even degrees is even.

Therefore the sum of all odd degrees must be even.

A sum of odd integers is even only if there are an even number of them. □

Incidence Counting

An incidence is a relationship between two types of objects.

Examples:

- students belonging to clubs,
- points lying on lines,
- vertices incident to edges,
- subsets containing elements.

Example

There are 10 clubs, each with 6 students. If 30 students each belong to exactly r clubs, find r.

Solution. Count student-club memberships.

Counting by clubs:

$$10 \cdot 6 = 60.$$

Counting by students:

$$30r.$$

Thus:

$$30r = 60,$$

so:

$$r = 2.$$

□

Vandermonde's Identity

Key Idea

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

Solution. Count committees of size k from a group of m boys and n girls.

Directly:

$$\binom{m+n}{k}.$$

Alternatively, choose i boys and $k-i$ girls:

$$\binom{m}{i}\binom{n}{k-i}.$$

Summing over all possible i gives:

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

Thus the identity follows. □

Hockey-Stick Identity

Key Idea

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Solution. Count ways to choose $r+1$ numbers from:

$$\{1, 2, \ldots, n+1\}.$$

Directly:

$$\binom{n+1}{r+1}.$$

Now count by the largest chosen number. If the largest chosen number is $k+1$, then the remaining r numbers must be chosen from:

$$\{1, 2, \ldots, k\}.$$

This can be done in:

$$\binom{k}{r}$$

ways.

Summing:

$$\sum_{k=r}^{n} \binom{k}{r}.$$

Thus:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

□

Advanced Worked Example Problems

Example

Problem 1. How many arrangements of the letters in TRIANGLE have no two vowels adjacent?

Solution. The vowels are:

$$A, E, I$$

and the consonants are:

$$T, R, N, G, L.$$

First arrange the 5 consonants:

$$5!.$$

They create 6 gaps:

$$\_C\_C\_C\_C\_C\_.$$

Choose 3 of these gaps for the vowels:

$$\binom{6}{3}.$$

Arrange the vowels:

$$3!.$$

Thus:

$$5!\binom{6}{3}3!.$$

□

Example

Problem 2. Count integers from 1 to 1000 not divisible by 2, 3, or 5.

Solution. First count those divisible by at least one of $2, 3, 5$.

From earlier PIE:

$$734$$

integers are divisible by at least one of them.

There are 1000 integers total.

Thus the answer is:

$$1000 - 734 = 266.$$

□

Example

Problem 3. Prove:
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Solution. Count ways to choose n objects from $2n$ objects split into two groups of size n.

Directly:
$$\binom{2n}{n}.$$

Alternatively, choose k objects from the first group and $n-k$ from the second:
$$\binom{n}{k}\binom{n}{n-k}.$$

Since:
$$\binom{n}{n-k} = \binom{n}{k},$$

the total is:
$$\sum_{k=0}^{n} \binom{n}{k}^2.$$

Thus:
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

□

Example

Problem 4. Prove:
$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}.$$

Solution. Count triples (S, a, b) where:
$$S \subseteq \{1, 2, \dots, n\}$$

and:
$$a, b \in S.$$

The pair (a, b) is ordered, and $a = b$ is allowed.

Method 1: Choose S first. If $|S| = k$, there are k^2 ordered choices for (a, b). This gives:
$$\sum_{k=0}^{n} k^2 \binom{n}{k}.$$

Method 2: Choose (a, b) first.

If $a = b$, there are n choices and 2^{n-1} choices for the rest of S.

If $a \neq b$, there are $n(n-1)$ choices and 2^{n-2} choices for the rest of S.
Thus:

$$n2^{n-1} + n(n-1)2^{n-2}.$$

Factor:

$$n2^{n-2}(2 + n - 1) = n(n+1)2^{n-2}.$$

□

Common Patterns

Pattern 1: Order Matters

Use permutations, factorials, or ordered choices.

Pattern 2: Order Does Not Matter

Use combinations.

Pattern 3: Objects Repeat

Divide by symmetry.

Pattern 4: At Least One

Try complementary counting or inclusion–exclusion.

Pattern 5: No Two Adjacent

Use gaps or complementary counting.

Pattern 6: Same Object Counted Two Ways

Use double counting.

Problem Set

Introductory Problems

1. How many binary strings of length 8 are there?
2. How many binary strings of length 8 contain exactly three 1's?
3. How many ways can 6 distinct books be arranged on a shelf?
4. How many arrangements are there of the letters in LEVEL?
5. How many committees of 4 can be chosen from 10 people?
6. How many three-digit numbers have distinct digits?
7. How many nonnegative integer solutions are there to $x + y + z = 10$?

8. How many positive integer solutions are there to $x + y + z = 10$?
9. Count integers from 1 to 100 divisible by 2 or 5.
10. Find the number of derangements of 3 objects.

Intermediate Problems

11. How many arrangements of 5 boys and 4 girls have no two girls adjacent?
12. How many ways can 8 people sit around a circular table?
13. Count arrangements of the letters in COMMITTEE.
14. How many integers from 1 to 1000 are divisible by 3 or 7?
15. How many integers from 1 to 1000 are divisible by none of $2, 3, 5$?
16. Prove Pascal's identity combinatorially.
17. Prove the hockey-stick identity combinatorially.
18. Prove Vandermonde's identity combinatorially.
19. Prove:
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$
20. Prove:
$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Advanced Problems

21. How many permutations of $1, 2, \ldots, 8$ have 1 appearing before 2 and 3 appearing before 4?
22. How many arrangements of the letters in MATHEMATICS are possible?
23. How many arrangements of the letters in TRIANGLE have no two vowels adjacent?
24. Find the number of derangements of 5 objects.
25. Count the number of onto functions from a 5-element set to a 3-element set.
26. Prove:
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$
27. Prove:
$$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}.$$
28. A graph has 20 vertices and 45 edges. Find the sum of all vertex degrees.

29. Prove that every finite graph has an even number of vertices with odd degree.
30. There are 12 clubs, each with 10 students. If each student belongs to exactly 3 clubs, how many students are there?
31. How many integer solutions to:
$$x_1 + x_2 + x_3 + x_4 = 25$$
satisfy $x_i \geq 2$ for all i?
32. How many nonnegative integer solutions to:
$$x + y + z = 20$$
satisfy $x \leq 5$?
33. Use inclusion–exclusion to count the number of integers from 1 to 500 divisible by at least one of $4, 6, 9$.
34. Count arrangements of A, B, C, D, E, F, G where A and B are adjacent but C and D are not adjacent.
35. Create your own double counting proof of a binomial identity.

Problem-Solving Guidance

For introductory problems, focus on deciding whether order matters and whether objects repeat.

For intermediate problems, expect combinations of techniques: permutations with restrictions, inclusion–exclusion, and basic double counting.

For advanced problems, the difficulty is usually recognizing the correct structure. A problem may appear to be about arrangements, but the solution may involve complementary counting. A problem may appear to require algebra, but the clean solution may be double counting.

Strategies for Olympiad Success

- Define the objects being counted before choosing a formula.
- Decide whether order matters.
- Check for repeated or identical objects.
- Look for symmetry and overcounting.
- Use complementary counting when the condition says “at least one” or “none.”
- Use inclusion–exclusion when cases overlap.
- Use gaps for no-adjacency restrictions.
- Use blocks for adjacency restrictions.
- Use double counting when an identity or incidence structure appears.

- Always verify that every object is counted exactly once.

Common Mistake

Many olympiad students lose points in combinatorics because they compute too early. First understand the structure; then calculate.

Section Strategy Summary

When facing a counting problem, try the following sequence:

1. Identify the objects.
2. Decide whether order matters.
3. Check whether objects are distinct or repeated.
4. Look for restrictions.
5. Decide whether direct counting or complementary counting is easier.
6. Check whether cases overlap.
7. If overlap exists, consider inclusion–exclusion.
8. If an identity appears, try double counting.
9. Verify that every valid object is counted exactly once.

Closing Thoughts

Even though combinatorics is an abstract mathematical subject, consistent problem solving can greatly develop the combinatorial ability. Two problems that look similar may require completely different approaches, while two problems that look unrelated may secretly have the same structure. Experience and persistence are crucial.