Polynomials on the AMC 10/12
Guide #1 – Roots, Vieta, and Transformations

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Contents

1 Introduction

Polynomials are possibly the most important algebraic topic on the AMC 10/12. While basic techniques like factoring and quadratic roots are covered in typical high school curriculum, competitive questions test deeper depth and breadth regarding these structures.
This handout focuses on polynomial roots including Vieta's formulas, reciprocal roots, and more.
Before looking over the solutions, it is suggested to give each problem a notable effort.

2 Fundamental Results

Before we dive into the flashy techniques, it is important to have a basic grasp over fundamental algebraic results.

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0.$$

By definition, the degree of P is n.

Fundamental Theorem of Algebra

Every nonconstant polynomial of degree n with complex coefficients has exactly n complex roots, counted with multiplicity.
Thus a degree-n polynomial can be written as

$$P(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n),$$

where $r_1, r_2, \ldots, r_n$ are its roots, including repeated roots.

For example,

$$x^3 - 3x^2 + 3x - 1 = (x - 1)^3$$

has three roots counted with multiplicity, even though all three are equal to 1.

Factor Theorem

For a polynomial $P(x)$,

$$P(r) = 0$$

if and only if

$$x - r$$

is a factor of $P(x)$.

Thus known roots immediately give polynomial factors, and known factors immediately give roots.

Example 2.1

Suppose $P(x)$ is a cubic polynomial satisfying

$$P(2) = P(-1) = P(4) = 0.$$

If the leading coefficient is 3, determine $P(x)$.

Solution. By the Factor Theorem, $x-2$, $x+1$, and $x-4$ are factors. Since P is cubic and has leading coefficient 3,

$$P(x) = 3(x-2)(x+1)(x-4).$$

Thus

$$\boxed{P(x) = 3(x-2)(x+1)(x-4)}.$$

□

Remainder Theorem

When a polynomial $P(x)$ is divided by $x-a$, the remainder is

$$P(a).$$

Equivalently,

$$P(x) = (x-a)Q(x) + P(a)$$

for some polynomial $Q(x)$.

Example 2.2

Find the remainder when

$$P(x) = x^9 - 3x^7 + 2x^4 - 5$$

is divided by $x-2$.

Solution. By the Remainder Theorem, the remainder is $P(2)$:

$$P(2) = 2^9 - 3(2^7) + 2(2^4) - 5 = 512 - 384 + 32 - 5 = \boxed{155}.$$

□

3 Polynomial Roots

Suppose

$$P(x) = a(x-r_1)(x-r_2)\cdots(x-r_n).$$

Evaluating at any number t gives

$$P(t) = a(t-r_1)(t-r_2)\cdots(t-r_n).$$

This can powerful for evaluating complex expressions without individually computing the components of the product.

Example 3.1

Let a, b, c, d be the roots of

$$P(x) = 2x^4 - 3x^3 - 5x^2 + 7x - 6.$$

Find

$$(a+2)(b+2)(c+2)(d+2).$$

Solution. By the root representation of the polynomial, it is shown that

$$P(x) = 2(x-a)(x-b)(x-c)(x-d),$$

we have

$$P(-2) = 2(a+2)(b+2)(c+2)(d+2).$$

Now

$$P(-2) = 32 + 24 - 20 - 14 - 6 = 16.$$

Therefore

$$(a+2)(b+2)(c+2)(d+2) = \boxed{8}.$$

□

Example 3.2

A monic cubic $P(x)$ has roots a, b, c. Suppose

$$P(1) = 8, \qquad P(-1) = -32.$$

Find

$$\frac{(a+1)(b+1)(c+1)}{(a-1)(b-1)(c-1)}.$$

Solution. Since P is monic,

$$P(-1) = -(a+1)(b+1)(c+1) = -32,$$

so the numerator is 32. Also,

$$P(1) = -(a-1)(b-1)(c-1) = 8,$$

so the denominator is -8. Therefore the required ratio is

$$\boxed{-4}.$$

□

4 Vieta's Relations

Vieta's relations connect the coefficients of a polynomial to symmetric expressions in its roots. Suppose

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has roots $r_1, r_2, \ldots, r_n$. By the Fundamental Theorem of Algebra,

$$P(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

Expanding this product and comparing coefficients shows that the coefficients of the polynomial encode the elementary symmetric sums of the roots.

Vieta's Relations

If

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has roots $r_1, r_2, \ldots, r_n$, then

$$r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n},$$

$$\sum_{1 \le i < j \le n} r_i r_j = \frac{a_{n-2}}{a_n},$$

with alternating signs continuing through the higher-order products. In particular,

$$r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}.$$

For a cubic with roots a, b, c, it is convenient to write

$$S = a + b + c, \qquad P = ab + bc + ca, \qquad Q = abc.$$

Vieta then allows many expressions involving a, b, c to be rewritten entirely in terms of S, P, and Q. For instance,

$$a^2 + b^2 + c^2 = S^2 - 2P,$$

while

$$(a + b)(b + c)(c + a) = SP - Q.$$

Similarly,

$$a^3 + b^3 + c^3 = S^3 - 3SP + 3Q.$$

These identities are especially useful when the roots themselves are difficult or impossible to write explicitly, but the problem asks for a symmetric expression involving them.

Example 4.1

Let a, b, c be the roots of

$$x^3 - 6x^2 + 9x - 2 = 0.$$

Find

$$a^2 + b^2 + c^2.$$

Solution. By Vieta, $a + b + c = 6$ and $ab + bc + ca = 9$. Therefore

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 36 - 18 = \boxed{18}.$$

□

5 Symmetric Expressions

An expression in the roots is called symmetric if its value does not change when the roots are permuted. Such expressions can often be rewritten entirely in terms of the quantities appearing in Vieta's relations.

For three variables,

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca),$$

and

$$(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc.$$

Also,

$$(a - b)^2 + (b - c)^2 + (c - a)^2$$

can be rewritten as

$$2\left(a^2 + b^2 + c^2 - ab - bc - ca\right).$$

Example 5.1

Let a, b, c be the roots of

$$x^3 - 7x^2 + 14x - 8 = 0.$$

Find

$$(a + b)(b + c)(c + a).$$

Solution. By Vieta,

$$a + b + c = 7, \qquad ab + bc + ca = 14, \qquad abc = 8.$$

Using

$$(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc,$$

the value is

$$7(14) - 8 = \boxed{90}.$$

□

Example 5.2

Let a, b, c be the roots of

$$x^3 - 7x^2 + 14x - 5 = 0.$$

Find

$$a^2b^2 + b^2c^2 + c^2a^2.$$

Solution. Let $u = ab$, $v = bc$, and $w = ca$. Then

$$u + v + w = 14,$$

while

$$uv + vw + wu = abc(a + b + c) = 5 \cdot 7 = 35.$$

Therefore

$$u^2 + v^2 + w^2 = (u + v + w)^2 - 2(uv + vw + wu) = 14^2 - 70 = \boxed{126}.$$

□

6 Reciprocal Roots

Suppose a, b, c are nonzero. Then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc},$$

and

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a + b + c}{abc}.$$

Thus Vieta's relations can be used directly on expressions involving reciprocals.

Example 6.1

Let a, b, c be the roots of

$$2x^3 - 5x^2 + 7x - 3 = 0.$$

Find

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Solution. By Vieta,

$$a + b + c = \frac{5}{2}, \qquad ab + bc + ca = \frac{7}{2}, \qquad abc = \frac{3}{2}.$$

Therefore

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{7/2}{3/2}=\frac{7}{3},$$

and

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{5/2}{3/2}=\frac{5}{3}.$$

Hence

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\left(\frac{7}{3}\right)^2-2\left(\frac{5}{3}\right)=\boxed{\frac{19}{9}}.$$

□

7 Transforming Roots

Suppose $P(x)$ has roots

$$r_1, r_2, \ldots, r_n.$$

If every root is shifted by the same constant, the polynomial can be transformed by replacing the variable.

If the desired roots are

$$r_1+k, \ldots, r_n+k,$$

then

$$Q(x)=P(x-k)$$

has exactly those roots.

Example 7.1

The roots of

$$P(x)=x^3-6x^2+5x+12$$

are a, b, c. Find the monic polynomial whose roots are

$$a+2, \qquad b+2, \qquad c+2.$$

Solution. The required polynomial is

$$Q(x)=P(x-2).$$

Thus

$$Q(x)=(x-2)^3-6(x-2)^2+5(x-2)+12,$$

which simplifies to

$$\boxed{x^3-12x^2+41x-30}.$$

□

7.1 Reciprocal Polynomials

If

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has nonzero roots $r_1, \ldots, r_n$, then

$$x^n P\left(\frac{1}{x}\right)$$

has roots

$$\frac{1}{r_1}, \ldots, \frac{1}{r_n}.$$

Its coefficients appear in reverse order:

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Example 7.2

The roots of

$$2x^3 - 3x^2 - 5x + 7 = 0$$

are a, b, c. Find a polynomial whose roots are

$$\frac{1}{a}, \qquad \frac{1}{b}, \qquad \frac{1}{c}.$$

Solution. Reverse the coefficients of the original polynomial. One suitable polynomial is

$$\boxed{7x^3 - 5x^2 - 3x + 2}.$$

□

8 Reciprocal and Palindromic Polynomials

A polynomial such as

$$x^4 + ax^3 + bx^2 + ax + 1$$

has coefficients that read the same in both directions. Dividing by x^2 gives

$$x^2 + \frac{1}{x^2} + a\left(x + \frac{1}{x}\right) + b.$$

Set

$$y = x + \frac{1}{x}.$$

Since

$$x^2 + \frac{1}{x^2} = y^2 - 2,$$

the quartic reduces to a quadratic in y.

Example 8.1

Solve

$$x^4 - 7x^3 + 14x^2 - 7x + 1 = 0.$$

Solution. Divide by x^2 and let

$$y = x + \frac{1}{x}.$$

Then

$$y^2 - 2 - 7y + 14 = 0,$$

so

$$y^2 - 7y + 12 = 0.$$

Thus $y = 3$ or $y = 4$. Solving

$$x + \frac{1}{x} = 3$$

and

$$x + \frac{1}{x} = 4$$

gives

$$\boxed{x = \frac{3 \pm \sqrt{5}}{2}, \quad 2 \pm \sqrt{3}}.$$

□

9 Problem Set

The following problems range from direct applications to more involved root transformations and symmetric expressions. Later problems are intended to be similar in difficulty to AMC 12 and Early AIME problems.

9.1 Introductory Problems

1. Let a, b, c be the roots of
$$x^3 - 7x^2 + 12x - 3 = 0.$$
Find $a^2 + b^2 + c^2$.

2. Let r and s be the roots of
$$2x^2 - 7x + 3 = 0.$$
Find
$$\frac{r}{s} + \frac{s}{r}.$$

3. Let a, b, c be the roots of
$$x^3 - 5x^2 + 8x - 4 = 0.$$
Find
$$(a+1)(b+1)(c+1).$$

4. Let a, b, c be the roots of
$$x^3 - 8x^2 + 18x - 7 = 0.$$
Find
$$(a+b)(b+c)(c+a).$$

5. Let a, b, c be the roots of
$$x^3 - 6x^2 + 8x - 3 = 0.$$
Find
$$(a-b)^2 + (b-c)^2 + (c-a)^2.$$

6. Let a, b, c be the nonzero roots of
$$2x^3 - 5x^2 + 7x - 4 = 0.$$
Find
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

7. Let a, b, c be the roots of
$$x^3 - 4x^2 + 6x - 2 = 0.$$
Find
$$a^3 + b^3 + c^3.$$

8. A monic quartic polynomial P has roots a, b, c, d. If
$$P(-2) = 35,$$
find
$$(a+2)(b+2)(c+2)(d+2).$$

9.2 Intermediate Problems

9. Let a, b, c be the roots of
$$x^3 - 6x^2 + 11x - 5 = 0.$$
Find
$$a^2b^2 + b^2c^2 + c^2a^2.$$

10. Let a, b, c be the roots of
$$2x^3 - 9x^2 + 10x - 3 = 0.$$
Find
$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

11. Let a, b, c be the roots of
$$x^3 - 5x^2 + 7x - 4 = 0.$$
Find
$$(a+b)^2 + (b+c)^2 + (c+a)^2.$$

12. The roots of
$$x^3 - 6x^2 + 9x - 1 = 0$$
are a, b, c. Find the monic polynomial whose roots are
$$a + 2, \qquad b + 2, \qquad c + 2.$$

13. The nonzero roots of
$$3x^3 - 7x^2 + 2x + 5 = 0$$
are a, b, c. Find a polynomial with integer coefficients whose roots are
$$\frac{1}{a}, \qquad \frac{1}{b}, \qquad \frac{1}{c}.$$

14. Let a, b, c be the roots of
$$x^3 - 7x^2 + 13x - 4 = 0.$$
Find
$$(a + b)^2(b + c)^2(c + a)^2.$$

15. A monic cubic polynomial has roots a, b, c satisfying
$$a + b + c = 8, \qquad a^2 + b^2 + c^2 = 30, \qquad abc = 5.$$
Find
$$(a + 1)(b + 1)(c + 1).$$

16. Suppose
$$x + \frac{1}{x} = 5.$$
Find
$$x^4 + \frac{1}{x^4}.$$

17. Find the sum of the real roots of
$$x^4 - 9x^3 + 20x^2 - 9x + 1 = 0.$$

18. Find all real roots of
$$x^4 - 6x^3 + 10x^2 - 6x + 1 = 0.$$

9.3 Advanced Problems

19. Let a, b, c be the roots of
$$x^3 - 4x^2 - 3x + 7 = 0.$$
Find
$$(a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2).$$

20. Let a, b, c be the nonzero roots of
$$x^3 - 6x^2 + 9x - 2 = 0.$$
Find
$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2.$$

21. Let a, b, c be the roots of
$$x^3 - 5x^2 + 4x + 3 = 0.$$
Find
$$(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2).$$

22. The roots of
$$2x^3 - 5x^2 + x + 4 = 0$$
are a, b, c. Find a cubic polynomial with integer coefficients whose roots are
$$\frac{a-1}{a+1}, \qquad \frac{b-1}{b+1}, \qquad \frac{c-1}{c+1}.$$

23. Suppose $x > 0$ and
$$x^3 + \frac{1}{x^3} = 110.$$
Find
$$x + \frac{1}{x}.$$

24. A monic cubic polynomial P has roots a, b, c and satisfies
$$a + b + c = 7, \qquad (a+b)(b+c)(c+a) = 31, \qquad abc = 4.$$
Find
$$ab + bc + ca.$$

25. Let a, b, c be the roots of
$$x^3 - 3x^2 - 6x + 2 = 0.$$
Find
$$(a^2 + b^2 + c^2)^2 - (ab + bc + ca)^2.$$

26. A monic polynomial P of degree 4 satisfies
$$P(1) = P(2) = P(3) = P(4) = 6.$$
If $P(0) = 30$, find $P(5)$.

27. Let a, b, c be the roots of
$$x^3 - 4x^2 + x + 1 = 0.$$
Given that
$$(a-b)(b-c)(c-a) > 0,$$
find
$$(a+b+c)(a-b)(b-c)(c-a).$$

9.4 Problems Adapted from Published Contests

28. *Adapted from 2013 AMC 12B, Problem 25.*

    A monic polynomial with integer coefficients has exactly four nonreal roots,

    $$a+bi, \qquad a-bi, \qquad c+di, \qquad c-di,$$

    where a,b,c,d are integers. Its constant term is 65. If

    $$a^2+b^2=5,$$

    find

    $$c^2+d^2.$$

29. *Adapted from 1984 AIME, Problem 15.*

    A monic quartic polynomial $F(t)$ has roots

    $$4, \qquad 16, \qquad 36, \qquad 64.$$

    Suppose its coefficient of t^3 can also be written as

    $$-\left(1+9+25+49+S\right).$$

    Find S.

30. Let the five roots of

    $$x^5-10x^4+Ax^3+Bx^2+Cx+D$$

    form an arithmetic progression. Find the middle root.

10 Solutions and Answers

1. By Vieta, $a+b+c=7$ and $ab+bc+ca=12$. Therefore

    $$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=49-24=\boxed{25}.$$

2. Vieta gives $r+s=\frac{7}{2}$ and $rs=\frac{3}{2}$. Hence

    $$\frac{r}{s}+\frac{s}{r}=\frac{r^2+s^2}{rs}=\frac{(r+s)^2-2rs}{rs}=\frac{\frac{49}{4}-3}{\frac{3}{2}}=\boxed{\frac{37}{6}}.$$

3. If $P(x)=x^3-5x^2+8x-4$, then

    $$P(-1)=(-1-a)(-1-b)(-1-c)=-(a+1)(b+1)(c+1).$$

    Since $P(-1)=-18$, the required product is $\boxed{18}$.

4. By Vieta, $a+b+c=8$, $ab+bc+ca=18$, and $abc=7$. Using

    $$(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc,$$

    the desired value is $8(18)-7=\boxed{137}$.

5. Vieta gives $a+b+c=6$ and $ab+bc+ca=8$. Thus
$$(a-b)^2+(b-c)^2+(c-a)^2 = 2\left((a+b+c)^2-3(ab+bc+ca)\right) = 2(36-24) = \boxed{24}.$$

6. By Vieta, $ab+bc+ca=\frac{7}{2}$ and $abc=2$. Therefore
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+bc+ca}{abc} = \boxed{\frac{7}{4}}.$$

7. Vieta gives $a+b+c=4$, $ab+bc+ca=6$, and $abc=2$. Hence
$$a^3+b^3+c^3 = (a+b+c)^3-3(a+b+c)(ab+bc+ca)+3abc = 64-72+6 = \boxed{-2}.$$

8. Because P is monic,
$$P(-2) = (-2-a)(-2-b)(-2-c)(-2-d) = (a+2)(b+2)(c+2)(d+2).$$
Thus the answer is $\boxed{35}$.

9. Vieta gives $a+b+c=6$, $ab+bc+ca=11$, and $abc=5$. Therefore
$$a^2b^2+b^2c^2+c^2a^2 = (ab+bc+ca)^2-2abc(a+b+c) = 121-60 = \boxed{61}.$$

10. Vieta gives
$$a+b+c=\frac{9}{2}, \qquad ab+bc+ca=5, \qquad abc=\frac{3}{2}.$$
Thus
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{10}{3}, \qquad \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=3,$$
and therefore
$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \left(\frac{10}{3}\right)^2-2(3) = \boxed{\frac{82}{9}}.$$

11. By Vieta, $a+b+c=5$ and $ab+bc+ca=7$, so
$$a^2+b^2+c^2 = 25-14 = 11.$$
Hence
$$(a+b)^2+(b+c)^2+(c+a)^2 = 2(a^2+b^2+c^2+ab+bc+ca) = 2(11+7) = \boxed{36}.$$

12. If $P(x)=x^3-6x^2+9x-1$, then replacing each root a by $a+2$ corresponds to
$$Q(x) = P(x-2).$$
Thus
$$Q(x) = (x-2)^3-6(x-2)^2+9(x-2)-1 = \boxed{x^3-12x^2+45x-51}.$$

13. Replacing x by $1/x$ gives
$$3x^{-3}-7x^{-2}+2x^{-1}+5=0.$$
Multiplying by x^3 gives
$$\boxed{5x^3+2x^2-7x+3}.$$

14. Using

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc,$$

we obtain

$$7(13) - 4 = 87.$$

Therefore

$$(a+b)^2(b+c)^2(c+a)^2 = 87^2 = \boxed{7569}.$$

15. Since

$$30 = (a+b+c)^2 - 2(ab+bc+ca),$$

we have $30 = 64 - 2(ab+bc+ca)$, giving $ab+bc+ca = 17$. Therefore

$$(a+1)(b+1)(c+1) = abc + (ab+bc+ca) + (a+b+c) + 1 = 5 + 17 + 8 + 1 = \boxed{31}.$$

16. From $x + \frac{1}{x} = 5$,

$$x^2 + \frac{1}{x^2} = 25 - 2 = 23.$$

Thus

$$x^4 + \frac{1}{x^4} = 23^2 - 2 = \boxed{527}.$$

17. Dividing the equation by x^2 and setting

$$y = x + \frac{1}{x}$$

gives

$$y^2 - 9y + 18 = 0.$$

Thus $y = 3$ or 6. Each value gives a reciprocal pair of real roots, and the sum of all four roots is therefore $\boxed{9}$.

18. Dividing by x^2 and setting $y = x + \frac{1}{x}$ gives

$$y^2 - 6y + 8 = 0,$$

so $y = 2$ or 4. The case $y = 2$ gives $x = 1$, while $y = 4$ gives $x = 2 \pm \sqrt{3}$. Thus the roots are

$$\boxed{1, 1, 2+\sqrt{3}, 2-\sqrt{3}}.$$

19. By Vieta, $a+b+c = 4$, $ab+bc+ca = -3$, and $abc = -7$. Hence

$$a^2 + b^2 + c^2 = 4^2 - 2(-3) = 22$$

and

$$a^2b^2 + b^2c^2 + c^2a^2 = (-3)^2 - 2(-7)(4) = 65.$$

The required value is

$$22^2 - 2(65) = \boxed{354}.$$

20. Set $u = 1/a$, $v = 1/b$, and $w = 1/c$. Vieta gives

$$u + v + w = \frac{ab + bc + ca}{abc} = \frac{9}{2}, \qquad uv + vw + wu = \frac{a + b + c}{abc} = 3.$$

Thus

$$u^2 + v^2 + w^2 = \left(\frac{9}{2}\right)^2 - 2(3) = \frac{57}{4}.$$

Therefore

$$(u - v)^2 + (v - w)^2 + (w - u)^2 = 2\left(\frac{57}{4} - 3\right) = \boxed{\frac{45}{2}}.$$

21. Vieta gives $a + b + c = 5$, $ab + bc + ca = 4$, and $abc = -3$. Thus

$$a^2 + b^2 + c^2 = 25 - 8 = 17$$

and

$$a^2b^2 + b^2c^2 + c^2a^2 = 4^2 - 2(-3)(5) = 46.$$

The required product is

$$17(46) = \boxed{782}.$$

22. Let

$$y = \frac{x - 1}{x + 1}, \qquad \text{so} \qquad x = \frac{1 + y}{1 - y}.$$

Substituting into $2x^3 - 5x^2 + x + 4 = 0$ and multiplying by $(1 - y)^3$ gives

$$2\left(2y^3 + 11y^2 - 6y + 1\right) = 0.$$

Hence one suitable polynomial is

$$\boxed{2y^3 + 11y^2 - 6y + 1}.$$

23. Let

$$y = x + \frac{1}{x}.$$

Then

$$x^3 + \frac{1}{x^3} = y^3 - 3y,$$

so $y^3 - 3y = 110$. Since $y = 5$ satisfies the equation and $x > 0$ gives $y \geq 2$, the answer is $\boxed{5}$.

24. Using

$$(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc,$$

we obtain

$$31 = 7(ab + bc + ca) - 4.$$

Therefore

$$ab + bc + ca = \boxed{5}.$$

25. Vieta gives $a + b + c = 3$ and $ab + bc + ca = -6$. Thus

$$a^2 + b^2 + c^2 = 3^2 - 2(-6) = 21,$$

and the required value is

$$21^2 - (-6)^2 = 441 - 36 = \boxed{405}.$$

26. Since $P(x) - 6$ is a monic quartic with roots $1, 2, 3, 4$,

$$P(x) - 6 = (x-1)(x-2)(x-3)(x-4).$$

Therefore

$$P(5) = 6 + (4)(3)(2)(1) = \boxed{30}.$$

27. The discriminant of

$$x^3 - 4x^2 + x + 1$$

is 169, so

$$((a-b)(b-c)(c-a))^2 = 169.$$

The given sign condition implies $(a-b)(b-c)(c-a) = 13$. Since $a+b+c = 4$, the desired value is

$$4(13) = \boxed{52}.$$

28. Because the polynomial has real coefficients, the nonreal roots occur in conjugate pairs. Their product is

$$(a+bi)(a-bi)(c+di)(c-di) = (a^2+b^2)(c^2+d^2).$$

Thus

$$5(c^2+d^2) = 65,$$

so

$$\boxed{13}.$$

29. By Vieta, the sum of the roots of F is

$$4 + 16 + 36 + 64 = 120.$$

Therefore the coefficient of t^3 is -120. Since

$$1 + 9 + 25 + 49 = 84,$$

we have $84 + S = 120$, giving

$$\boxed{36}.$$

30. Write the five roots as

$$m - 2d, \qquad m - d, \qquad m, \qquad m + d, \qquad m + 2d.$$

Their sum is $5m$. By Vieta, the sum of the roots is 10, so

$$5m = 10.$$

Hence the middle root is

$$\boxed{2}.$$

‘‘‘