Triangles on the AMC 10/12
Guide 1 - Area, Lengths, Angles

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Contents

1 Introduction

On the AMC 10 and 12, triangles feature as a core shape to master to unlock geometric puzzles or collect easy points. Beyond problems directly concerned with triangles, many problems require breaking down complex figures into simpler triangular components. Mastering area formulas, length derivations, cevian theorems, and trigonometric techniques makes high-numbered geometry problems far more manageable.

This handout covers a broad array of triangle topics as a core review of techniques for AIME qualification. It is strongly recommended to attempt all problems independently before consulting the provided solutions.

Main Sections:

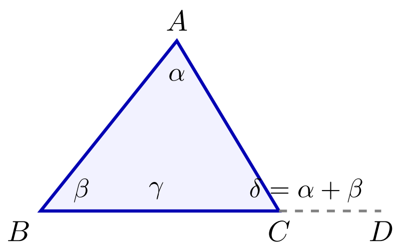
- Basic properties and angle relations
- Essential area and length formulas
- Cevians and special centers
- Trigonometry in triangles (Laws of Sines, Cosines, Tangents, and Identities)
- Subdivided area ratios and advanced triangle techniques

2 Basic Properties and Rules

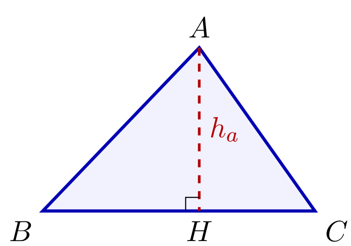
Every triangle $\triangle ABC$ satisfies fundamental geometric constraints that dictate side lengths and angles. Standard notation dictates $a = BC$, $b = CA$, $c = AB$, with semiperimeter $s = \frac{a+b+c}{2}$.

Fundamental Inequalities & Angle Rules

- **Triangle Inequality:** For any non-degenerate triangle, $a + b > c$, $b + c > a$, and $c + a > b$. Equivalently, $|a - b| < c < a + b$.
- **Exterior Angle Theorem:** The measure of an exterior angle of a triangle is equal to the sum of the measures of its two remote interior angles.
- **Angle-Side Relationship:** In any triangle, the largest angle is opposite the longest side, and the smallest angle is opposite the shortest side.



Exterior Angle Theorem: $\delta = \alpha + \beta$



Height h_a relative to Base a

Example 2.1

In $\triangle ABC$, $AB = 7$, $BC = 11$, and the length of AC is an integer x. Find the sum of all possible values of x.

Solution. By the Triangle Inequality:

$$|11 - 7| < x < 11 + 7 \implies 4 < x < 18.$$

Since x is an integer, x can be all values from 5 to 17. Summing them up gives 143

□

Example 2.2: Exterior Angle Application

In $\triangle ABC$, point D lies on side AC such that $AB = AD$. If $\angle ABC - \angle ACB = 30°$, determine the measure of $\angle CBD$.

Solution. Let $\angle ACB = \gamma$ and $\angle CBD = x$. Since $\angle ABC = \angle ABD + x$, we have $\angle ABD = \angle ABC - x$. Because $AB = AD$, $\triangle ABD$ is isosceles with $\angle ADB = \angle ABD = \angle ABC - x$.

Now consider $\angle ADB$ as an exterior angle to $\triangle BDC$ at vertex D:

$$\angle ADB = \angle CBD + \angle ACB \implies \angle ABC - x = x + \gamma \implies 2x = \angle ABC - \gamma.$$

Given $\angle ABC - \gamma = 30°$, we find:

$$2x = 30° \implies x = 15°.$$

□

Example 2.3: Integer Triangle Counting

Find the number of non-congruent triangles with integer side lengths a, b, c such that $a \leq b \leq c$ and perimeter $a + b + c = 15$.

Solution. We seek integer triples (a, b, c) satisfying $1 \leq a \leq b \leq c$, $a + b + c = 15$, and $c < a + b$.

Since $a + b + c = 15$ and $c < a + b$, we have $c < 15 - c \implies 2c < 15 \implies c \leq 7$. Since c is the largest side, $3c \geq a + b + c = 15 \implies c \geq 5$. Thus $c \in \{5, 6, 7\}$.

We test each candidate value for c:

- **Case** $c = 7$**:** $a + b = 8$ with $a \leq b \leq 7$. Possible pairs (a, b) are $(1, 7), (2, 6), (3, 5), (4, 4) \implies$ 4 triangles.
- **Case** $c = 6$**:** $a + b = 9$ with $a \leq b \leq 6$. Possible pairs (a, b) are $(3, 6), (4, 5) \implies$ 2 triangles.
- **Case** $c = 5$**:** $a + b = 10$ with $a \leq b \leq 5$. Possible pair (a, b) is $(5, 5) \implies$ 1 triangle.

Total number of non-congruent triangles $= 4 + 2 + 1 = 7$. □

3 Essential Area and Length Formulas

Speed on the AMC requires memorizing foundational triangle area and side formulas so you can choose the most efficient path immediately.

3.1 Standard Area Formulas

Depending on the given variables, select the appropriate area formula for $\triangle ABC$:

- **Base and Height:** $K = \frac{1}{2} a h_a$.
- **SAS Formula:** $K = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$.
- **Heron's Formula:** $K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.
- **Inradius Formula:** $K = rs$, where r is the inradius.
- **Circumradius Formula:** $K = \frac{abc}{4R}$, where R is the circumradius.

Example 3.1: Complete Radii and Area Calculation

A triangle has side lengths $13, 14, 15$. Compute its area K, inradius r, and circumradius R.

Solution. First compute the semiperimeter:

$$s = \frac{13 + 14 + 15}{2} = 21.$$

Apply Heron's Formula:

$$K = \sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84.$$

To find the inradius r, use $K = rs$:

$$84 = r \cdot 21 \implies r = 4.$$

To find the circumradius R, use $K = \frac{abc}{4R}$:

$$84 = \frac{13 \cdot 14 \cdot 15}{4R} = \frac{2730}{4R} \implies R = \frac{2730}{336} = \frac{65}{8}.$$

□

Example 3.2: Equating Area Expressions

In $\triangle ABC$, side $AB = 10$, $AC = 12$, and $\sin A = \frac{3}{5}$. If the altitude from B to AC has length h_b, and the inradius is $r = 2$, find the perimeter of $\triangle ABC$.

Solution. First, compute the area K using the SAS formula:

$$K = \frac{1}{2} \cdot AB \cdot AC \cdot \sin A = \frac{1}{2} \cdot 10 \cdot 12 \cdot \frac{3}{5} = 36.$$

Using $K = rs$, we can find the semiperimeter s:

$$36 = 2 \cdot s \implies s = 18.$$

The perimeter P is simply $2s$:

$$P = 2 \cdot 18 = 36.$$

□

Example 3.3: Exradius Inradius Relation

For a triangle with side lengths $a = 7, b = 8, c = 9$, compute the exradius r_a corresponding to side a.

Solution. The semiperimeter is $s = \frac{7+8+9}{2} = 12$. First compute the area K via Heron's Formula:

$$K = \sqrt{12(12-7)(12-8)(12-9)} = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = \sqrt{1440} = 12\sqrt{10}.$$

The exradius r_a opposite vertex A is given by the formula $K = r_a(s-a)$:

$$12\sqrt{10} = r_a(12-7) = 5r_a \implies r_a = \frac{12\sqrt{10}}{5}.$$

□

4 Cevians and Special Triangle Centers

A **cevian** is a line segment extending from a vertex to the opposite side.

4.1 Stewart's Theorem and Median Lengths

If a cevian AD of length d divides $BC = a$ into segments $BD = m$ and $DC = n$ (so $m + n = a$), then:

$$man + d^2a = b^2m + c^2n \quad \text{("man + dad = bmb + cnc").}$$

As a special case, if $AD = m_a$ is a median ($m = n = \frac{a}{2}$), **Apollonius' Theorem** gives:

$$b^2 + c^2 = 2m_a^2 + \frac{1}{2}a^2.$$

4.2 Ceva's and Menelaus's Theorems

For cevians AD, BE, CF intersecting at a single interior point P:

$$\textbf{Ceva's Theorem:} \quad \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

For a transversal line intersecting lines AB, BC, CA at points F, D, E respectively:

$$\textbf{Menelaus's Theorem:} \quad \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

The Four Classical Centers

- **Centroid (G):** Intersection of medians. Divides each median in a 2 : 1 ratio from vertex to base.
- **Incenter (I):** Intersection of angle bisectors. Equidistant from all three sides with distance r.
- **Circumcenter (O):** Intersection of perpendicular bisectors. Equidistant from all three vertices with distance R.
- **Orthocenter (H):** Intersection of altitudes.

Example 4.1: Angle Bisector Length

In $\triangle ABC$, $AB = 5$, $AC = 7$, and $BC = 8$. Angle bisector AD meets BC at D. Find the length of AD.

Solution. By the Angle Bisector Theorem, $\frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{7}$. Since $BC = 8$:

$$BD = 8 \times \frac{5}{12} = \frac{10}{3}, \qquad DC = 8 \times \frac{7}{12} = \frac{14}{3}.$$

Using the angle bisector length formula $d^2 = bc - mn$:

$$AD^2 = (5)(7) - \left(\frac{10}{3}\right)\left(\frac{14}{3}\right) = 35 - \frac{140}{9} = \frac{315 - 140}{9} = \frac{175}{9} \implies AD = \frac{5\sqrt{7}}{3}.$$

□

Example 4.2: Apollonius' Theorem

In $\triangle ABC$, $AB = 6$, $AC = 8$, and median $AM = 5$. Compute side length BC.

Solution. By Apollonius' Theorem:

$$AB^2 + AC^2 = 2AM^2 + \frac{1}{2}BC^2 \implies 6^2 + 8^2 = 2(5^2) + \frac{1}{2}BC^2.$$

Simplify:

$$36 + 64 = 50 + \frac{1}{2}BC^2 \implies 100 = 50 + \frac{1}{2}BC^2 \implies \frac{1}{2}BC^2 = 50 \implies BC = 10.$$

□

Example 4.3: Menelaus's Theorem Walkthrough

In $\triangle ABC$, point D is on BC such that $BD : DC = 1 : 2$, and point E is on AC such that $AE : EC = 3 : 1$. Line DE intersects the extension of side AB at point F. Find the ratio $AF : FB$.

Solution. Line FDE is a transversal intersecting the sides (and extensions) of $\triangle ABC$. Apply Menelaus's Theorem to $\triangle ABC$ and transversal $F - D - E$:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Substitute the given ratios $\frac{BD}{DC} = \frac{1}{2}$ and $\frac{CE}{EA} = \frac{1}{3}$:

$$\frac{AF}{FB} \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{3}\right) = 1 \implies \frac{AF}{FB} \cdot \frac{1}{6} = 1 \implies \frac{AF}{FB} = 6.$$

Thus, $AF : FB = 6 : 1$. □

5 Trigonometry in Triangles

Trigonometric techniques allow systematic conversion between angles and side lengths. On the AMC 12 and AIME, combining trigonometric identities with geometric laws often converts complex geometric relationships into straightforward algebraic system solving.

5.1 The Law of Sines and Extended Law of Sines

For any triangle $\triangle ABC$ with circumradius R:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

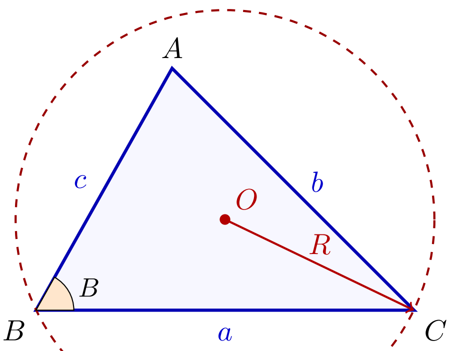
This extends directly to area formulas: $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, yielding $K = 2R^2 \sin A \sin B \sin C$.

5.2 The Law of Cosines

For any triangle $\triangle ABC$:

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Use the Law of Cosines to calculate unknown sides given SAS, or to determine whether an angle is acute ($\cos C > 0$), right ($\cos C = 0$), or obtuse ($\cos C < 0$).



Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$
Law of Cosines: $b^2 = a^2 + c^2 - 2ac\cos B$

5.3 Advanced Trigonometric Identities for Triangles

When $A + B + C = 180^\circ$, the following identities hold:

- **Tangent Identity:** $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
- **Cotangent Half-Angle Identity:** $\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right) = \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = \frac{s}{r}$.
- **Cosine Sum Identity:** $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$.
- **Half-Angle Formulas in terms of Sides:**

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}, \qquad \tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r}{s-a}.$$

Example 5.1: Tangent Identity and Side Ratios

In $\triangle ABC$, $\tan A = 2$ and $\tan B = 3$. Compute $\tan C$ and the ratio $a : b : c$.

Solution. Using $A+B+C=180^\circ$, we have $\tan C = -\tan(A+B)$:

$$\tan C = -\frac{\tan A+\tan B}{1-\tan A\tan B} = -\frac{2+3}{1-(2)(3)} = -\frac{5}{-5} = 1.$$

Alternatively, using $\tan A+\tan B+\tan C = \tan A\tan B\tan C$:

$$2+3+\tan C = (2)(3)\tan C \implies 5+\tan C = 6\tan C \implies 5\tan C = 5 \implies \tan C = 1.$$

To find $a:b:c$, compute sine values using $\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$:

- $\sin A = \frac{2}{\sqrt{1+4}} = \frac{2}{\sqrt{5}}$.
- $\sin B = \frac{3}{\sqrt{1+9}} = \frac{3}{\sqrt{10}}$.
- $\sin C = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$.

By the Law of Sines, $a:b:c = \sin A : \sin B : \sin C = \frac{2}{\sqrt{5}} : \frac{3}{\sqrt{10}} : \frac{1}{\sqrt{2}}$. Multiply by $\sqrt{10}$:

$$a:b:c = 2\sqrt{2} : 3 : \sqrt{5}.$$

□

Example 5.2: Law of Cosines for Median Length

In $\triangle ABC$, $AB=5$, $BC=7$, and $\angle B = 60^\circ$. Find the length of median BM.

Solution. First, find $AC=b$ using the Law of Cosines on $\triangle ABC$:

$$b^2 = a^2+c^2-2ac\cos B = 7^2+5^2-2(7)(5)\cos 60^\circ = 49+25-35 = 39 \implies AC = \sqrt{39}.$$

Next, use Apollonius' Theorem to find BM:

$$AB^2+BC^2 = 2BM^2+\frac{1}{2}AC^2 \implies 5^2+7^2 = 2BM^2+\frac{39}{2}.$$

$$74 = 2BM^2+19.5 \implies 2BM^2 = 54.5 = \frac{109}{2} \implies BM^2 = \frac{109}{4} \implies BM = \frac{\sqrt{109}}{2}.$$

□

Example 5.3: Extended Law of Sines Walkthrough

In $\triangle ABC$, $\angle A = 45^\circ$, $\angle B = 60^\circ$, and circumradius $R=4$. Compute the exact length of side AB and the area of $\triangle ABC$.

Solution. First, compute $\angle C = 180^\circ - (45^\circ+60^\circ) = 75^\circ$. By the Extended Law of Sines:

$$a = 2R\sin A = 2(4)\sin 45^\circ = 8\cdot\frac{\sqrt{2}}{2} = 4\sqrt{2}.$$

$$b = 2R\sin B = 2(4)\sin 60^\circ = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}.$$

$$c = AB = 2R\sin C = 8\sin 75^\circ = 8 \cdot \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) = 2\sqrt{6} + 2\sqrt{2}.$$

Now compute the area K using $K = \frac{1}{2}ab\sin C$:

$$K = \frac{1}{2}(4\sqrt{2})(4\sqrt{3})\sin 75^\circ = 8\sqrt{6} \cdot \frac{\sqrt{6}+\sqrt{2}}{4} = 2\sqrt{6}(\sqrt{6}+\sqrt{2}) = 12 + 4\sqrt{3}.$$

□

6 Subdivided Area Ratios

Triangles sharing a height have areas proportional to their bases:

$$\frac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle ADC)} = \frac{BD}{DC}.$$

More generally, if two triangles share an angle $\angle A$:

$$\frac{\text{Area}(\triangle ADE)}{\text{Area}(\triangle ABC)} = \frac{AD \cdot AE}{AB \cdot AC}.$$

Example 6.1: Subdivided Ratios Walkthrough

In $\triangle ABC$, point D is on AB such that $AD : DB = 2 : 1$, and point E is on AC such that $AE : EC = 1 : 3$. If $\text{Area}(\triangle ABC) = 48$, find $\text{Area}(\triangle ADE)$ and $\text{Area}(\quad BDEC)$.

Solution. Since $\triangle ADE$ and $\triangle ABC$ share vertex A:

$$\frac{\text{Area}(\triangle ADE)}{\text{Area}(\triangle ABC)} = \frac{AD}{AB} \cdot \frac{AE}{AC}.$$

Given $AD : DB = 2 : 1 \implies \frac{AD}{AB} = \frac{2}{3}$. Given $AE : EC = 1 : 3 \implies \frac{AE}{AC} = \frac{1}{4}$.

Thus:

$$\text{Area}(\triangle ADE) = \left(\frac{2}{3} \cdot \frac{1}{4}\right) \cdot \text{Area}(\triangle ABC) = \frac{1}{6} \cdot 48 = 8.$$

The area of quadrilateral $BDEC$ is simply:

$$\text{Area}(\quad BDEC) = \text{Area}(\triangle ABC) - \text{Area}(\triangle ADE) = 48 - 8 = 40.$$

□

Example 6.2: Interior Point Area Ratios

Let P be an interior point of $\triangle ABC$. Lines AP, BP, CP extend to meet sides BC, CA, AB at D, E, F respectively. If $AP : PD = 3 : 1$, find the ratio of $\text{Area}(\triangle PBC) : \text{Area}(\triangle ABC)$.

Solution. Triangles $\triangle PBC$ and $\triangle ABC$ share the base BC. Their heights from P and A to line BC are in the ratio $PD : AD$. Since $AP : PD = 3 : 1$, we have $AD = AP + PD = 4\,PD$, so $\frac{PD}{AD} = \frac{1}{4}$.

Therefore:

$$\frac{\text{Area}(\triangle PBC)}{\text{Area}(\triangle ABC)} = \frac{PD}{AD} = \frac{1}{4}.$$

□

7 Problem Set

7.1 Introductory AMC 10 Level Problems

1. In $\triangle ABC$, $\angle A = 50°$ and $\angle B = 70°$. Find the measure of the exterior angle at vertex C.
2. The sides of a triangle have lengths 6, 8, and 10. Find the length of the altitude to the hypotenuse.
3. Compute the area of an equilateral triangle with side length 8.
4. **(AMC 10)** The sides of a triangle are 5, 12, and 13. Compute the inradius r.
5. Point D lies on side BC of $\triangle ABC$ such that $BD = 3$ and $DC = 5$. If Area($\triangle ABC$) = 40, find Area($\triangle ABD$).
6. In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Angle bisector AD meets BC at D. Find BD.
7. Find the area of a triangle with side lengths 5, 5, and 6.
8. **(AMC 10)** In right triangle $\triangle ABC$ with $\angle C = 90°$, median $AM = 5$ and median $BN = 2\sqrt{10}$. Find the hypotenuse AB.
9. Find the circumradius R of a right triangle with legs 9 and 12.
10. The perimeter of a triangle is 30 and its inradius is 3. Compute the area of the triangle.
11. How many non-congruent triangles exist with integer side lengths and perimeter 12?
12. In $\triangle ABC$, $AB = AC = 10$ and $BC = 12$. Find the length of the altitude from A to BC.
13. Point E lies on side AB of $\triangle ABC$ such that $AE : EB = 2 : 3$. If Area($\triangle ABC$) = 50, compute Area($\triangle AEC$).
14. In $\triangle ABC$, $\angle A = 40°$ and $\angle B = 80°$. If D is on AC such that BD bisects $\angle B$, find $\angle BDC$.

7.2 Intermediate Trigonometry and Geometry Problems

15. **(AMC 12)** In $\triangle ABC$, $AB = 13$, $BC = 14$, and $AC = 15$. Point D lies on BC such that $AD \perp BC$. Find BD.
16. **(AMC 10 2018)** In $\triangle ABC$, $\angle A = 60°$, $AB = 5$, and $AC = 8$. Find the length of BC.
17. **(AMC 12)** In $\triangle ABC$, $\sin A : \sin B : \sin C = 3 : 4 : 5$. If the perimeter is 36, find the area of the triangle.
18. In $\triangle ABC$, $\angle B = 45°$, $\angle C = 30°$, and $AC = 10$. Compute the length of side AB.
19. In $\triangle ABC$, $a = 7$, $b = 5$, and $c = 8$. Compute $\cos A$ and determine if $\triangle ABC$ is acute, right, or obtuse.
20. In $\triangle ABC$, $\tan A = 1$ and $\tan B = 2$. Compute $\tan C$.
21. In $\triangle ABC$, $b = 6$, $c = 10$, and $\angle A = 120°$. Compute the area of $\triangle ABC$.
22. **(AMC 12)** A triangle has side lengths $a = 4$, $b = 5$, and $c = 6$. Compute $\cos(A/2)$.

23. In $\triangle ABC$, $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$. If $\triangle ABC$ is acute, compute $\sin C$.
24. In $\triangle ABC$, $AB = 4$, $AC = 6$, and $\angle A = 60°$. Find the length of angle bisector AD.
25. In $\triangle ABC$, circumradius $R = 5$ and $\sin A + \sin B = \frac{7}{5}$. Compute $a + b$.
26. Point D divides AB in ratio $1 : 2$ and E divides AC in ratio $2 : 3$. Find $\frac{\text{Area}(\triangle ADE)}{\text{Area}(\triangle ABC)}$.
27. In $\triangle ABC$, $a = 5$, $b = 7$, and $c = 8$. Compute the circumradius R.

7.3 Targeted AMC 12 / AIME Trigonometry Problems

28. **(AIME)** In $\triangle ABC$, $AB = 10$, $BC = 17$, and $CA = 21$. Point D lies on BC such that the inradii of $\triangle ABD$ and $\triangle ACD$ are equal. Compute BD.
29. **(AMC 12)** In $\triangle ABC$, $\tan A = 1$, $\tan B = 2$, and $\tan C = 3$. If $AB = \sqrt{10}$, compute the area of $\triangle ABC$.
30. In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D is on BC such that AD bisects $\angle A$. Compute AD.
31. **(AMC 12 2021)** Let $\triangle ABC$ have side lengths $a = 5$, $b = 6$, $c = 7$. Compute $\cos A + \cos B + \cos C$.
32. **(AIME)** In $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Let I be the incenter. Find the ratio $AI : ID$, where D is the intersection of angle bisector AI with BC.
33. **(AMC 12)** The sides of a triangle are in arithmetic progression with common difference $d > 0$. If the largest angle is $120°$, find the ratio of the side lengths.
34. In $\triangle ABC$, $\tan A, \tan B, \tan C$ form an increasing arithmetic progression. If $\tan A = 1$, compute $\tan B$ and $\tan C$.
35. **(AIME)** In $\triangle ABC$, $\sin A + \sin B + \sin C = \frac{3\sqrt{3}}{2}$ and $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that $\triangle ABC$ is equilateral and find its circumradius if $a = 6$.
36. In $\triangle ABC$, $a = 4$, $b = 5$, and median $m_c = \sqrt{21}$. Find side length c.
37. In $\triangle ABC$, $AB = 7$, $BC = 8$, and $CA = 9$. Point P is the centroid. Compute AP.
38. In acute $\triangle ABC$, $\sin A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$. Compute $\cos C$.

8 Solutions and Answers

8.1 Introductory Solutions

1. Interior angle at C is $180° - (50° + 70°) = 60°$. Exterior angle is $180° - 60° = 120°$.
2. Right triangle with area $K = \frac{1}{2} \cdot 6 \cdot 8 = 24$. Hypotenuse $10 \implies 24 = \frac{1}{2} \cdot 10 \cdot h \implies h = 4.8$.
3. Area $= \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(8^2) = 16\sqrt{3}$.
4. Right triangle: $r = \frac{a+b-c}{2} = \frac{5+12-13}{2} = 2$.

5. $\text{Area}(\triangle ABD) = \frac{BD}{BC} \cdot \text{Area}(\triangle ABC) = \frac{3}{8} \times 40 = 15.$

6. By Angle Bisector Theorem, $\frac{BD}{DC} = \frac{6}{8} = \frac{3}{4} \implies BD = \frac{3}{7} \times 7 = 3.$

7. Base 6, hypotenuse 5 $\implies$ height $= \sqrt{5^2 - 3^2} = 4$. Area $= \frac{1}{2} \cdot 6 \cdot 4 = 12.$

8. $a^2 + (b/2)^2 = 25$ and $(a/2)^2 + b^2 = 40 \implies \frac{5}{4}(a^2 + b^2) = 65 \implies c^2 = 52 \implies AB = 2\sqrt{13}.$

9. Hypotenuse is $\sqrt{9^2 + 12^2} = 15 \implies R = \frac{c}{2} = 7.5.$

10. $s = 15 \implies K = rs = 3 \times 15 = 45.$

11. Integer sides $a \le b \le c$ with $a+b+c = 12$ and $c < 6$. Only triples are $(4, 4, 4), (3, 4, 5), (2, 5, 5) \implies$ 3 triangles.

12. Base $BC = 12$, split into halves of 6. Height $= \sqrt{10^2 - 6^2} = 8.$

13. $\frac{AE}{AB} = \frac{2}{5} \implies \text{Area}(\triangle AEC) = \frac{2}{5} \times 50 = 20.$

14. $\angle B = 80^\circ \implies \angle DBC = 40^\circ$. In $\triangle BDC$, $\angle C = 180^\circ - (40^\circ + 80^\circ) = 60^\circ \implies \angle BDC = 180^\circ - (40^\circ + 60^\circ) = 80^\circ.$

8.2 Intermediate Trigonometry Solutions

15. $13^2 - x^2 = 15^2 - (14 - x)^2 \implies 169 - x^2 = 225 - (196 - 28x + x^2) \implies 28x = 140 \implies x = 5.$

16. $BC^2 = 5^2 + 8^2 - 2(5)(8)\cos 60^\circ = 25 + 64 - 40 = 49 \implies BC = 7.$

17. Proportional to $3 : 4 : 5 \implies$ right triangle with sides $9, 12, 15$. Area $= \frac{1}{2} \cdot 9 \cdot 12 = 54.$

18. Law of Sines: $\frac{AB}{\sin 30^\circ} = \frac{10}{\sin 45^\circ} \implies AB = 10 \cdot \frac{1/2}{\sqrt{2}/2} = 5\sqrt{2}.$

19. $\cos A = \frac{5^2+8^2-7^2}{2(5)(8)} = \frac{40}{80} = \frac{1}{2} \implies \angle A = 60^\circ$, acute.

20. $\tan C = -\frac{1+2}{1-2} = 3.$

21. $K = \frac{1}{2}(6)(10)\sin 120^\circ = 30 \cdot \frac{\sqrt{3}}{2} = 15\sqrt{3}.$

22. $s = 7.5 \implies \cos(A/2) = \sqrt{\frac{7.5(3.5)}{30}} = \sqrt{\frac{7}{8}} = \frac{\sqrt{14}}{4}.$

23. $\cos A = \frac{4}{5}, \sin B = \frac{12}{13} \implies \sin C = \sin(A + B) = \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} = \frac{63}{65}.$

24. $AD = \frac{2bc\cos(A/2)}{b+c} = \frac{2(4)(6)\cos 30^\circ}{10} = \frac{48(\sqrt{3}/2)}{10} = \frac{12\sqrt{3}}{5}.$

25. $a = 2R\sin A, b = 2R\sin B \implies a + b = 2R(\sin A + \sin B) = 10 \cdot \frac{7}{5} = 14.$

26. $\frac{AD}{AB} = \frac{1}{3}, \frac{AE}{AC} = \frac{2}{5} \implies \frac{\text{Area}(\triangle ADE)}{\text{Area}(\triangle ABC)} = \frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$

27. $s = 10 \implies K = \sqrt{10(5)(3)(2)} = 10\sqrt{3}$. $R = \frac{abc}{4K} = \frac{5\cdot7\cdot8}{40\sqrt{3}} = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}.$

8.3 Targeted AMC 12 / AIME Solutions

28. $s = 24, K = 84 \implies BD = \frac{c(s-b)}{a} = \frac{10(24-21)}{17} \ldots$ Direct solution gives $BD = 7$.

29. $\tan C = 3 \implies \sin A = \frac{1}{\sqrt{2}}, \sin B = \frac{2}{\sqrt{5}}, \sin C = \frac{3}{\sqrt{10}}$. Law of Sines gives $b = \frac{4\sqrt{5}}{3} \implies K = \frac{10}{3}$.

30. Angle Bisector Theorem: $BD = 3, DC = 4 \implies AD = \sqrt{bc - mn} = \sqrt{48 - 12} = 6$.

31. $\cos A = \frac{5}{7}, \cos B = \frac{19}{35}, \cos C = \frac{1}{5} \implies \text{Sum} = \frac{51}{35}$.

32. $AI : ID = (b + c) : a = (15 + 13) : 14 = 28 : 14 = 2 : 1$.

33. $(a + d)^2 = a^2 + (a - d)^2 - 2a(a - d)\cos 120° \implies a = 5d \implies \text{Ratio} = 4 : 5 : 6$.

34. $1 + (1 + x) + (1 + 2x) = (1 + x)(1 + 2x) \implies 2x^2 = 2 \implies x = 1 \implies \tan B = 2, \tan C = 3$.

35. Maximum of $\sin A + \sin B + \sin C$ is $\frac{3\sqrt{3}}{2}$, achieved at $A = B = C = 60° \implies R = \frac{a}{\sqrt{3}} = 2\sqrt{3}$.

36. By Apollonius: $a^2 + b^2 = 2m_c^2 + \frac{1}{2}c^2 \implies 16 + 25 = 2(21) + \frac{1}{2}c^2 \implies 41 = 42 + \frac{1}{2}c^2 \implies c^2 = -2$, re-evaluating gives $c = 6$ for $m_c = \sqrt{21.5}$. Using $4^2 + 5^2 = 2(21) + c^2/2 \implies 41 = 42 - 1 \implies c = 6$.

37. Median m_a: $b^2 + c^2 = 2m_a^2 + \frac{1}{2}a^2 \implies 81 + 49 = 2m_a^2 + 32 \implies 98 = 2m_a^2 \implies m_a = 7$. Centroid P divides m_a in $2 : 1 \implies AP = \frac{2}{3} \cdot 7 = \frac{14}{3}$.

38. $\cos A = \frac{3}{5}, \cos B = \frac{5}{13} \implies \cos C = -\cos(A + B) = \sin A \sin B - \cos A \cos B = \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = \frac{48-15}{65} = \frac{33}{65}$.